DARLEY.

POPULAR GEOMETRY.

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A SYSTEM

OF

POPULAR GEOMETRY.

A SYSTEM

OF

POPULAR GEOMETRY;

CONTAINING

IN A FEW LESSONS SO MUCH OF

THE ELEMENTS OF EUCLID

AS IS

NECESSARY AND SUFFICIENT FOR A RIGHT UNDERSTANDING OF EVERY

ART AND SCIENCE

IN ITS LEADING TRUTHS AND GENERAL PRINCIPLES.

BY GEORGE DARLEY, A. B.

THE FOURTH EDITION.

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ADVERTISEMENT

TO THE

SECOND EDITION.

It is far more creditable to the public than to us, that a Second Edition of our Geometry has been so soon called for. Encouragement given to those who labour in facilitating the road to knowledge is but an involuntary effect of proportional ardour in those who pursue it. To promote the dissemination of mathematical reasoning, or rather a *taste* for mathematical reasoning, amongst the several classes of society, was the object of our publication, and this object appears to have been fully attained. Nothing shall be wanting on our part to keep alive the spirit of inquiry, and to cherish it on to greater things than it now dares contemplate even in imagination. This we can speak of as yet but darkly; the reader may be assured, however, that if he, with the moderate exertion of patience and industry which we require, is content to ascend through the scale of works prepared for him, he will reach a height as much beyond his present conceptions as his present abilities.

We have been still more gratified by private communications on the subject of our Work than by the public patronage of it, inasmuch as they denote that personal interest in the study which we were solicitous to create. It would be impossible and unnecessary for us to answer

individually the numerous letters with which we have been favoured; but we beg leave to return our sincerest thanks to all our correspondents, the suggestions and wishes of whom we have complied with in our Second Edition, as far as it was practicable or advisable. We can truly declare, that in editing the present Series of Works, we have much more regard for the public good than our own mathematical reputation. Suggestions and objections were therefore, and will always be, acceptable; we feel no reluctance whatever to surrender our own opinion and preferences, if we can thereby accommodate our Treatises to the wants and wishes of those for whom they are compiled. It is in the power of few to become illustrious, but of every one to become useful; we are certainly not of the former class, but hope we may be of the latter.

The present edition of our Geometry has been carefully revised, and one or two oversights in the preceding corrected. Some of our Correspondents, habituated probably to the older system, complain of our having omitted the Axioms. To satisfy these, we subjoin each as it occurs, in a foot-note.

Upon the Doctrine of Ratio a fuller explanatory note is given; demonstrating its *adequacy* as well as simplicity. We did not give this explanation in our First Edition from an unwillingness to perplex a beginner, and lengthen our book with what then appeared needless. We thought that our readers would have been content with the general insight afforded them upon this subject, but are happy to find ourselves mistaken.

In order to render our work complete we have prefixed a table of Contents, which it will be often sufficient to consult when a reference is given, instead of examining the text itself for the Article or Definition specified.

The arrangement of the present Edition is in nowise

different from the first, except that we have substituted a more serviceable element for one which is of little use in our system (ART. 18, 1st ED.) This enables us also to simplify the demonstration of ART. 20, &c. Of the suppressed element, which is the objectionable one in Euclid, those who desire it may see, in the section of Useful Results, a complete demonstration from our principles, that given in our first Edition being insufficient. It was suggested that the proofs of certain very obvious propositions (such as ARTS. 33, 34, PART I., ARTS. 48, 49, PART II., &c.) were unnecessary. In order to shorten and simplify our Elements we have merely preserved the *enunciations* of these Articles in the text, transferring the *proofs* to the Notes, which may be consulted by the reader.

An Author's satisfaction in his own work is not always a just criterion of the applause it will gain with the public. *Our* satisfaction in those Treatises we now publish will, it is confessed, mainly depend on the approbation of our readers, inasmuch as the real merit of such books is proportionate to their utility. We should have as little reason for self-congratulation in having written a scientifical treatise (however admirable) which was not read, as an artist in constructing a patent machine which was not handled: the present edition of our Geometry will, we hope, secure us in the continuance of public favour, this being the only solid proof of our having deserved it.

POSTSCRIPT TO FOURTH EDITION.

The strictures upon Euclid in our Preface and Notes are, principally, as every scientific reader will be aware, those of able mathematicians, not our own. Whenever we found a necessity to deviate from that admirable work, we thought it but a due mark of respect to give our reasons for so doing. This motive should not be misconstrued.

PREFACE.

The necessity for a work, somewhat on the plan of the present, is the best excuse for an imperfect one, and the best recommendation of a good one. There is a Geometry (if it may be so called) for children, and a Geometry for professed mathematicians; but there is no Geometry for the public at large. We have nothing between Pinnock and Euclid. A Geometry as far beyond the one as behind the other, is the greatest desideratum in the world of literature.

Knowledge, as it exists now in the public mind, and as it tends to exist hereafter, requires such a work; in order to render that which is already gained more satisfactory, and that which is to be gained more accessible, than either can be without the aid of Geometry. This is the foundation of all scientific knowledge, as the study of it is the surest step to all Advancement in Learning of the philosophical kind. Without geometry may be obtained information of *facts*, or knowledge *by rote*, which is the sort now chiefly extant among us: But it is only by means of geometry that a scientifical *knowledge of things* can be acquired, in which we are now generally deficient.

A popular work upon almost every science has appeared: it is the design of the present to furnish a System of Geometry in like manner adapted for public use. There are three classes of persons for whom it is especially if not solely intended: Youth at public and private Schools; Persons whose Education has been neglected; Artists and Mechanics. To the first it will be an introduction to Euclid; to the second, a private and easy course of Elementary

Mathematics; to the third, a Manual containing all the geometrical principles on which the knowledge and practice of their several Arts are founded. We could point out a *fourth* class for whom our system of Geometry is peculiarly adapted: but custom, and illiberal prepossession against the ability of Ladies for studies of this nature, will probably ever prevent their deriving any advantage from a Science, whose very object is to *strengthen* the understanding.

As to the first class: It is well known by those who have the instruction of Youth, that the elements of Euclid is a work by much too large and difficult to constitute a proper school-book. Ere opened, its size begets fear, and when opened its abstruseness engenders disgust. Instead of rendering the principles of Geometry as familiar as possible, it renders them as abstract; not from an affectation of mystery, but from an anxiety for refinement. However desirable this may be in a philosophic point of view, it is highly inconvenient for the purposes of instruction. Beautiful as is the temple of Science, it lies so remote and inaccessible, that we must smooth and shorten the path to it, if we expect young and wavering dispositions to attempt the journey. The first object, therefore, of our Treatise is, to supply an *Introduction to Euclid*, for the use of Schools, whereby Youth may be gradually familiarised with the notions of Geometry, and after learning half the Elements with ease in our volume, those who are destined to the University may learn the whole of them without difficulty in larger ones. The labours of acquirement, and instruction, will be thus respectively and equally diminished.

As to the second class: Many persons whose education has been, from want of time, inclination, or resources, neglected, in after years have the will and power to remove their deficiency. It is too late with such to begin the scholastic course of education, but the more limited and private one, furnished by scientific yet popular Compendiums, is

within their ability. Likewise, there is a number of persons whose education, though far from having been neglected, was not of such a kind as the extraordinary advancement of knowledge, within the last few years, demands from those of their age or condition. If they have any acquaintance with the Sciences, it is superficial and altogether inconclusive. Many interesting truths which they meet with in their daily reading are unintelligible to them, many agreeable literary pursuits are shut against them. To this class of persons, also, a series of works like those described would be invaluable. But as Geometry is the key to all the Arts and Sciences, a popular treatise upon it is indispensable to those who would acquire the latter. That such a work should follow, instead of precede, the elementary treatises which have been published on Astronomy, &c. will not appear surprising when the abstract nature of the Science is considered. Our second object, therefore, is to furnish a *Substitute for Euclid*, from which persons whose education has been neglected, or restricted, may obtain as much geometrical knowledge as is sufficient, but not superfluous, in their circumstances.

As to the third class: Reasons similar to the above render Simson's Euclid (the only edition used in this country) totally unfit for the Artist or Mechanic, independent of its high price. Yet for no class of persons is the knowledge of Geometry more requisite. Every mechanical profession involves some principle of Geometry, and according as the Science is well understood so will be the profession, *ceteris paribus*. Every mechanical invention and improvement is founded on a principle of Geometry, and according as the Science is ready to the mind, so will be the inventions. It is true that professions have succeeded and inventions originated with those who were ignorant of Geometry; but this happened not *because* of their ignorance, but *in spite* of it. Where theory can direct practice, and practice suggest

to theory, there is the whole man brought to bear; and what neither could do separately, both will do conjointly. Education has spread too far among this most useful and respectable class of people to leave our assertion either unintelligible or doubtful. It is therefore our final object to publish for their use a *Compendium of Euclid*, brief to suit their time, easy to suit their apprehensions, and cheap to suit their purses.

To accomplish this threefold object, the same course is sufficient. First, the principles of the Science must be rendered as familiar, and brought as near to our commonest ideas as possible. For this purpose we have omitted all technical terms which could be dispensed with; we have given popular illustrations of some things which were abstract; and we have simplified many doctrines which were perplexed.

The next step should be to render the demonstrations of propositions as plain for the mind, and as brief for the memory, as possible. For this purpose we have made several alterations: Instead of notes of reference, which distract the mind, we have introduced the Articles themselves referred to; instead of accumulating many theorems under one head, as is frequently done in Euclid, we have separated and proved them distinctly. Where the proofs in Euclid appeared difficult, unsatisfactory, or prolix*, we have substituted easier, clearer, and shorter ones; and introduced other improvements of which the old methods were evidently susceptible.

Our last endeavour should be to reduce the Elements of Geometry, not only to their simplest, but to their shortest form. This was with us a principal object: conceiving it as unwise, as it is unphilosophical, to introduce as Elements of the Science propositions which are not such; thereby oppressing the reader's mind and memory with superfluous

* As in Propositions 5, 29, 26, Book I., &c.

matters, and destroying the uniformity of the system by a capricious intermixture of *principles* with *results*. For this last and best purpose we have excluded from the text, 1st. All such propositions as are not, rightly speaking—Elements of the Science; 2d. All such propositions as are unnecessary to the general reader of the Arts and Sciences. In many instances, likewise, we have contrived to dispense with auxiliary propositions, which are merely such, and have no farther influence*. Finally, instead of demonstrating general theorems from their particular cases, or after them†, our system has enabled us to deduce the latter immediately from the former, as is at once most natural and most simple.

In order to promote the joint object still further, we have likewise made the following arrangement:

Propositions are of two kinds, Theorems and Problems. A Theorem asserts that such a result will take place on such a condition; as, that—"if a triangle have two equal legs, it will also have two equal angles." A problem requires something to be done; such as—"to describe an equal-sided triangle on a given finite straight line." Some writers contend that the latter class of propositions are unnecessary, as in demonstrating it is only requisite that the thing should be *supposed* done, and that it is not requisite to do it actually. But, independent of the practical use of Problems, their exclusion from theory is dangerous, if not disallowable, for this reason: That we might, in such event, often assume that as practicable which was not so, and thus ground our demonstrations on a false premise. By doing a Problem actually, we prove that it *can* be done; nor is there any other method of being sure that we may suppose it in a demonstration. But as there are no pro-

* Such as Prop. 7th, Book I. This, which is called in Euclid a *Proposition*, is in truth only a *Lemma*.

† As is done by Euclid, Book I. Prop. 16, 17, 32, &c.

blems in our Elements of Geometry but such as are manifestly practicable—such as dividing a line or angle into two equal parts,—they are omitted in the text for the sake of brevity, but preserved in the page for the sake of accuracy. A double line separates the problems from the theorems; and they are so arranged that every problem assumed in a theorem precedes that theorem, while every theorem on which a problem depends precedes that problem. So that both theorems and problems may be read together as in Euclid, or each series will form a study by itself.

The limited nature of our work allows us to dispense with many Definitions, Axioms, &c., which are necessary in larger treatises. With respect to those Definitions and Axioms which we do make use of, we have preferred introducing them into our text and demonstrations, according as they are wanted; instead of loading the memory and revolting the taste with a number of barren truisms, before their utility is perceivable. Besides, the latter, formed as they are by rather a painful process of abstraction, will be much more easily acknowledged by the reader in the particular cases where they occur, than in the general form of independent propositions.

Having described the general features of our publication, we have yet to explain some particular ones. Those who have read the Elements which go under the name of Euclid, are aware that his Doctrine of Parallels, as it now stands, is unintelligible to most readers, and unsatisfactory to all. Some change was absolutely necessary to be made in it, if we would adapt our treatise for general comprehension. So long as Euclid's* doctrine of parallels forms, at least in its present state, the groundwork of the Science, Geometry

* It is by no means certain that the Elements of Euclid were composed by Euclid. Many authors seem to have had a share in them; so that it is unjust to accuse Euclid of errors attributable perhaps to others. But we use the name of Euclid for brevity.

can never be made a popular study. When such a monstrous assumption as the XIIth Axiom continues to stand in the very threshold of elementary mathematics, it will for ever repel common understandings, and for ever confine the study to those who have genius or perseverance much greater than ordinary. The philosopher or the professed scholar may be incited or obliged to overleap this broken step in his advance to knowledge; but it will be an eternal stop to the general reader, who, unless he finds the way plain, has no such motive to undertake it as the regular student. In our endeavour to render this doctrine more simple and conclusive, or at least more palatable, we have discovered a method which seems to attain, not only these objects, but another scarcely less important, namely, abbreviation. By our method of parallels, several propositions are demonstrable in a much shorter way than by the other; several abortions of propositions (as the 16th and 17th, B. I. Euclid, for instance) are avoided; several indirect proofs replaced by direct ones. The facility afforded by our doctrine of parallels to contract the length of our proofs and the size of our whole work, is sufficient to make us adop it, were it even, in point of strictness and intelligibility, no less objectionable than that in common use. We cannot help but flatter ourselves, however, that we have provided at least, for the general reader, a method of parallels, which by its simplicity, clearness, and satisfactoriness, will encourage him in the pursuit of Mathematical Science, instead of deterring him from it.

It is as far from our wish as it is from our ability to depreciate the Elements of Euclid; this work alone has been of more signal use to the world than all its other uninspired writings put together. That it has such great excellency is however no proof that it is without fault; that it is so good a book is no proof that it might not be made a better. In one point, that which we have just mentioned, the Doctrine *of* Parallels, it is confessedly imperfect. There

is another respect in which it is obviously capable of melioration: its arrangement. It appears singular enough that a work whose professed object is to exhibit a model of irrefragable demonstrative reasoning in union with the most perfect systematical arrangement, so as to be rather an object of envy than of emulation with other sciences,—should be erroneous at its very outset in reasoning, and defective in arrangement. But to say this is no more than to repeat a very ancient discovery,—that every thing human is imperfect. We should endeavour to fortify the Elements of Euclid where it is vulnerable, and to rectify it where it is out of rule, without dwelling on its faults, beyond what is proper and just, for the sake of displaying our own microscopic sagacity.

Geometry has a twofold use: to enlarge the knowledge and to improve the reason. Nor is it easy to say in which province it is most beneficial. If on the one hand it ministers to the practical comforts of life, by its influence over the various arts by which they are procured, on the other hand it elevates us higher in the scale of rational beings, and thus serves to enhance our intellectual pleasures. The latter is perhaps the nobler object, and therefore should be most sedulously promoted. It is however in this that the Elements of Euclid, though better calculated than any other book to advance both purposes, is chiefly defective; not, we repeat, that it does not do much, but that it does not do more, which it might be made to do easily. The reason is to be improved thus: by exercising the mind in *strict* demonstration, and in *systematic* demonstration. By the one it is preserved from fallacy, by the other from irregularity. Reasoning has two perfections; for there is a beauty as well as a strength of reasoning, and both should unite that the reasoning should be consummate. We obtain strength by strict demonstration, beauty by systematic demonstration.

If therefore the Elements of Euclid be so well adapted to improve the mind in both these ways, and if this capacity be one of its chief merits, it follows that the more this aptitude is promoted, this capacity augmented, the better for Human Reason. The principle of strict and systematic demonstration adopted in that work is the best ever devised by man to exalt himself as a rational being; why therefore is it not to be pushed as far as it can go? To have invented this admirable system was the glory of the Ancients; to have left it unimproved is the shame of the Moderns. Prejudice in favour of the Elements of Euclid has blinded us to its defects, while our very admiration of it should have incited us to render it still more worthy of that feeling*. In the unpretending little Treatise we now offer to the public, it cannot be *our* design to attempt perfectionating the Elements of Euclid; we leave that to abler heads and more ambitious hearts. Our work is in truth no more than the *elements* of the Elements of Euclid. But as far as it extends into that celebrated production, it endeavours to accomplish the above object; namely, to preserve inviolate strictness of demonstration in each proposition, and systematic demonstration in the whole series. This, which is commonly thought to be the essential principle of Euclid's work, is by no means there always carried into practice.

With regard to strictness of demonstration we have already adverted to Euclid's doctrine of Parallels, and our substitution of one which will perhaps be allowed as sufficiently clear and satisfactory. Of certain propositions loosely demonstrated by Euclid, his coadjutors, or commentators, we have given legitimate proofs; and supplied other propositions unaccountably omitted in the Elements. These changes comprise all which our treatise professes to

* This was the effect which NEWTON's admiration for Ancient Geometry had upon *his* mind: he improved their ***Method of Exhaustions*** into his own ***Method of Fluxions***.

have accomplished in the way of rendering the elements of Geometry more strict and conclusive. More has been done with respect to systematic demonstration, or arrangement of demonstrations. The arrangement of Euclid's work is not always good, and is often much the contrary. Amongst other examples, in Book I., the 32d proposition offends against the rules of systematic demonstration, by being inserted unnecessarily and unnaturally between two sets of propositions about parallelism. Book III. transgresses the same rule in the same way, but more inexcusably: there are first four propositions concerning the circle and straight line, then two about meeting circles, then three more about the circle and straight line again, then four more about meeting circles again, and still *again* more about the circle and straight line! Thus, without the smallest necessity the chain of demonstration is broken *four times* in a few pages. Such an arrangement looks more like a wilful transgression of system than a gross neglect of it. In Book VI. the propositions seem for the most part to be arranged by no fixed rule whatever, except that vague one of not placing the consequent before the antecedent. But in the Elements of Euclid there is likewise another species of offence committed against the rule of systematic demonstration; not only are the propositions ill-arranged, but they are often altogether superfluous. Now it is the beauty as well as the use of such a work, to be neither deficient nor redundant. From a solid foundation it should ascend step by step, each being the support of the following, till the last be gained. Every deviation from unity and simplicity of design is no less an error than a blemish. By a neglect of this rule the very best Geometries now extant are rendered perplexed and impure. In the present elementary work it has been our chief object to preserve the second rule of systematic demonstration; namely, to exclude every proposition which is not absolutely necessary, either as a

progressive step, or a final one. By this means we seem to have attained all the symmetrical regularity and chasteness of demonstration which this Science should present to its readers.

But as there are many propositions introduced by Euclid or his commentators, which although irrelevant to his subject, are nevertheless of considerable use in the Arts and Sciences, we have annexed to our work a section in which all such theorems and problems are contained, so that no reference to any other treatise may be necessary.

In this manner have we compiled a brief System of Geometry for popular use. It contains all of the Science which is necessary for the general reader, and nothing more. With the assistance of it alone he may attain a clear and competent knowledge of all the mathematical and physical branches of learning, in their elementary principles, their most useful truths, and most interesting results. For the want of such a little work as this, the unscholastic reader is completely shut out from a knowledge of those sublime and beautiful truths scattered over the sciences, from the magnificent wonders of Astronomy, the elegant details of Optics, and the ingenious discoveries of Mechanical Philosophy. These and other studies, far more entertaining, not to speak of their instructiveness, than the most fanciful poem or romance, are now open to him who is willing to cultivate his reason at the same time that he gratifies his curiosity.

We have but a few observations to subjoin. Our work is divided into three PARTS. PART I. is nearly identical with the first Book of Euclid, except that some useless propositions are retrenched, some necessary ones supplied, the arrangement of all systematically regulated, and the Doctrine of Parallels explained on somewhat different principles. PART II. contains all the propositions of Book II. and III. of Euclid which will be found necessary. In this part the

doctrine of the Circle precedes instead of following (as it does in Euclid) the doctrine of rectangles, which latter is rather the more difficult of comprehension. There is no good reason for the existing arrangement of those Books in Euclid, the last three propositions of Book III. which depend on Book II., being not elements, but results of elements, and therefore supernumerary. On the contrary, that Book which is made the third in Euclid is more elementary, and more used in elementary science, than that which is made the second. Also, by annexing the doctrine of the Circle to that of the Triangle, the Artist, the Mechanic, and the practical man will have, in that portion alone ofthe wor k, all that he will perhaps desire to know of the Science. PART III. comprises as much of the Doctrine of Proportion as is necessary and sufficient for the general reader to know: by a knowledge of this Part, the sciences of Algebra, Trigonometry, and Astronomy, &c. are to be approached and understood with as much ease as satisfaction. At the end of Part III. is placed the Section of Theorems and Problems, which were excluded from the text for their irrelevancy, but which are collected here for their utility.

Thus are all the useful principles and propositions of Euclid collected in a small, unexpensive volume; the obscure departments of the science made intelligible to common understandings; and the elements of Geometry so arranged as to unite systematic regularity with rigorous exactness of demonstration. The whole is divided into LESSONS, each of which may be mastered in a few hours, and all in a few weeks, by a diligent exertion of moderate abilities.

A COMPARATIVE TABLE, showing what Propositions of Euclid correspond with our Articles, is annexed, for the reader's convenience.

PRELIMINARY DIRECTIONS.

GEOMETRY is unlike every other study in this, that it contains no one section, paragraph, or sentence, more important than another. It follows from this, that to comprehend it perfectly, every section and paragraph, as far as the student proceeds,—and every *sentence* therein,—must be read, understood, and remembered. It may perhaps be said of Geometry that the more time we spend on it at first, the more shall we save at last. If the reader be careful to pass over nothing which he does not apprehend, and to forget nothing which he has apprehended, his advance, although slow at first, will be always sure, and swift at last. On the contrary, if he proceeds too fast in the beginning, it will be long before he reaches the end. Both these consequences arise from the nature of the Science; in it, each paragraph depends on a preceding one, and therefore will be wholly unintelligible, unless the other be not only understood but remembered. The first direction therefore that the reader should observe is: *to read every paragraph until he understands it fully, and to repeat it until he remembers the substance accurately.*

The second direction is: after having mastered every ARTICLE as given in the treatise, to attempt it *without the book*. First, let the student draw a figure for himself, *like* to that in the book, both as to position, shape, size, and letters; and let him apply *off-book* the demonstration, which he has studied, to his own figure, until he can prove the ARTICLE as it was proved in the work. Secondly, let him draw figures which vary in all the particulars of posi-

tion, shape, size, and letters from that in the book, as much as is possible under the conditions of the Article; and let him apply off-book the demonstration to these separately, until he has familiarised himself completely not only with the demonstrations, but with the *principle* of the demonstration.

The above directions regard the student in general; there are others which respect the particular classes for whom our work is designed, *viz.*

The principal text of the work contains *Definitions*, ARTICLES, and *Observations*. Those who wish to understand the *theory alone* of geometry—to arrive at those geometrical truths, merely, which are necessary for the other Sciences—will confine themselves to this part.

A short series of PROBLEMS, which comprise the practical part of the work, is divided by a double line from the preceding part. Those whose views are directed *solely* to *practice*, and who wish to acquire the geometrical elements of it, will read *this* series (leaving out the demonstrations). Previously, however, they must read the few introductory pages 1, 2, 3, down to ARTICLE 1, exclusive.

Both Series must be read by those who desire to obtain a knowledge of Geometry as well theoretical as practical. For such, an asterisk is placed, wherever a Problem, or Problems, are necessary to be read,—referring to the series at the bottom of the page.

ERRATA.

Page 84 line 3 from bottom, for C F, read C E.
— — last line, insert *given* before *ratio*.
— 86 line 8 before the word *This*, for D F, read E F.
— 88 line 11 from bottom, for A B G, read B A G.

CONTENTS.

Definition I. A rectilineal (or right-lined) ***Angle*** is that which is formed by two right lines meeting at the same point, but not lying in the same right line, 2.

Def. II. ***Figures*** are bounded portions of space, 3.

Def. III. A ***plane*** figure is a plane surface bounded by one or more lines, ibid.

Def. IV. A plane ***rectilineal*** figure is a plane surface bounded by ***right*** lines, ibid.

Def. V. A rectilineal triangle is a plane figure bounded by ***three*** right lines, ibid.

Def. VI. If one right line, standing upon another, make the adjacent angles equal to one another, each of these angles is called a ***right*** angle, and the right line which stands upon the other is called a ***perpendicular*** to it, 12.

Def. VII. An angle greater than a right angle is called an ***obtuse*** angle; and an angle less than a right angle is called an ***acute*** angle, 13.

Def. VIII. If two right lines cut each other so as to form ***four*** angles, each opposite pair are called ***Vertically*** opposite angles, 16.

Def. IX. Two right lines are said to be equally distant from one another when any two points whatsoever in the one not the greater, and any two equally remote points in the other, being taken, the right lines

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Prob. II. From a given point to draw a right line equal to a given finite right line, 6.

Prob. III. From the greater of two given right lines to cut off a part equal to the less, 7.

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Prob. VI. To draw a perpendicular to a given right line, from a given point without it, 11.

Prob. VII. To draw a perpendicular to a given right line, from a point in it, 12.

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Prob. IX. Through a given point outside a given right line to draw a right line parallel to the given one, 19.

which join each opposite pair of points towards the same hand are equal to each other, 16.

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Def. XI. A *parallelogram* is a four-sided rectilineal figure, each pair of whose opposite sides are parallel, 22.

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Def. XIV. A *Circle* is a plane figure bounded by one line, such, that all right lines drawn from it to one and the same point are equal to each other, 41.

Def. XV. In a circle the bounding line is called the *Circumference*, and the point to which the equal lines are drawn from the circumference is called the *Centre*, ibid.

Def. XVI. Any right line terminated both ways in a circle is called a *Chord* of the circle, or of the arch it cuts off, ibid.

Def. XVII. A right line which meets a circle in two points, but is not terminated in both, is called a *Secant*, 44.

Def. XVIII. A right line drawn from the centre of a circle and terminated in the circumference, is called a *Radius*, ibid.

Def. XIX. A right line drawn through the centre of a circle and terminated both ways in the circumference, is called a *Diameter*, ibid.

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Def. XXI. Right lines are said to be *equally distant* from the centre

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Prob. X. On a given right line to describe a square, 28.

Prob. XI. To find the centre of a given circle, 42.

Prob. XII. From a given point without a given circle, to draw a right line which shall be a tangent to the circle, 45.

Prob. XIII. A segment of a circle being given, to describe the circle of which it is the segment, 54.

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Prob. XV. To divide a given finite right line into any number of equal parts, 75.

Prob. XVI. To divide a right line into two parts which shall have the same ratio as the two parts of a given divided line, 84.

Prob. XVII. To cut off from a given right line a part which shall have a certain given ratio to the whole line, ibid.

Prob. XVIII. To find a fourth proportional to three given finite lines, 85.

Prob. XIX. To find a third proportional to two given finite right lines, ibid.

Prob. XX. To find a mean proportional between any two given finite right lines, 86.

of a circle when the perpendiculars drawn to them from the centre are equal. And one right line is said to be farther from the centre than another when the perpendicular on the former is greater than that on the latter, 47.

Def. XXII. If from a point outside a circle two tangents be drawn to the circle, and the points of contact be joined by a right line, that part of the circumference lying within the triangle thus formed is called the *Convex*, and that part of the circumference lying without this triangle is called the *Concave* part of the circumference, with respect to the given point, 51.

Def. XXIII. A rectilineal figure is said to be *inscribed* in a circle when the vertices of all its angles are situated in the circumference of that circle, 53.

Def. XXIV. If any geometrical figure or magnitude be divided into two or more parts, these parts are called *Segments*, 54.

Def. XXV. When an angle has its vertex in an arch of a circle, and its sides terminated in the extremities of that arch, this angle is called the *Angle in a segment* of the circle, 55.

Def. XXVI. The segments into which a circle is divided by its diameter are called *Semicircles*, 56.

Def. XXVII. When one circle meets another in two points, it is said to *cut* it, 58.

Def. XXVIII. Circles which meet in but one point are said to *touch* each other, 59.

Def. XXIX. The point where two circles touch is called the *point of contact*, ibid.

Def. XXX. Equal circles are those which, if applied centre to centre, would exactly coincide with each other, 60.

Def. XXXI. A *Rectangle* is a right-angled parallelogram, 64.

Def. XXXII. When one magnitude exactly equals another magnitude added to itself an integral number of times, the greater magnitude is called a *Multiple* of the lesser; and the lesser a *Submultiple* of the greater, 71.

Def. XXXIII. When two multiples contain their respective submultiples exactly an *equal* number of times, they are called *Equi-multiples*. And when two submultiples are contained in their respective multiples exactly an equal number of times they are called *Equi-submultiples*, ibid.

Prob. XXI. Given three finite lines, of which any two are greater than the third, to construct a triangle of which the sides shall be respectively equal to the given lines, 96.

Prob. XXII. To find a square equal to the sum of two squares, 97.

Prob. XXIII. To find a square equal to the difference of two squares, 98.

Prob. XXIV. To draw a tangent at any given point in the circumference of a circle, 106.

Prob. XXV. To divide a given right line into any number of parts which shall have the same ratio to each other as the parts of another given divided right line, 113.

Prob. XXVI. To make a square equal to a given rectangle, 114.

b

Def. XXXIV. *Ratio* is the mutual relation of two quantities of the same kind to each other, with respect to magnitude, 73.

Def. XXXV. Two quantities are said to have the same ratio to each other as two other quantities have, when every submultiple of the first quantity is contained in the second the same integral number of times that an *equi*-submultiple of the third quantity is contained in the fourth, 74.

Def. XXXVI. When a certain part of one magnitude has the same ratio to a certain part of another magnitude, as a second part *of this other* has to a second part of the first,—the proportion subsisting between these four parts is called, for brevity's sake, *reciprocal* proportion, 77.

Def. XXXVII. When two parallelograms have an angle in the one equal to an angle in the other, they are called *equi-angular* parallelograms, ibid.

Def. XXXVIII. If there be a series of magnitudes of the same kind, in which the 1st has to the 2d the same ratio as the 2d has to the 3d; and the 2d has to the 3d the same ratio as the 3d has to the 4th; and the 3d has to the 4th the same ratio as the 4th has to the 5th; and so on—then, these magnitudes are said to be in *continued* proportion, 80.

Def. XXXIX. When three magnitudes are proportionals, the first is said to have to the third a ratio *duplicate* of that which it has to the second, 91.

Def. XL. If through a given point in the diagonal of a parallelogram two right lines be drawn parallel respectively to two adjacent sides of the parallelogram, so as to make four parallelograms,—then, those two through which the diagonal runs are called *parallelograms about the diagonal*, and the remaining two are called *complements* of the parallelograms about the diagonal, 100.

Axiom 1. Two right lines cannot completely enclose a space, 4.

Ax. 2. Magnitudes which coincide exactly with each other are equal, 5.

Ax. 3. Things which are equal to the same thing are equal to each other, 6.

Ax. 4. If from equal things we take away equal things, respectively, the remainders will be equal, ibid.

Ax. 5. The whole is greater than its part, 9.

Ax. 6. If to equal things we add equal things, respectively, the wholes will be equal, 11.

Ax. 7. The halves of equal things are equal, 24.

Ax. 8. The doubles of equal things are equal, 26.

Ax. 9. If from unequal things we take equal things, the remainders are unequal, 49.

Article 1. If there be two triangles which have two sides of the one equal, respectively, to two sides of the other; and likewise the angles contained by those sides equal to one another:—then, the bases or third sides of these triangles are also equal, 3.

Art. 2, In such triangles as above described, those angles at the bases which are opposite to equal sides, are respectively equal, 5.

Art. 3. Such triangles as above described are equal, in every respect, to each other, ibid.

Art. 4. A triangle which has two of its sides equal, has also its angles, opposite the equal sides, equal, 8.

Art. 5. A triangle which has two of its angles equal, has also its sides opposite the equal angles equal, 9.

Art. 6. If there be two triangles which have all the sides of the one equal, respectively, to all the sides of the other,—then the angles, which in these triangles are opposite to equal sides, are also equal, ibid.

Art. 7. Such triangles as are described in the preceding Article are equal, in every respect, to each other, 11.

Art. 8. All right angles are equal, 13.

Art. 9. When a right line meeting another right line makes angles with it, these angles are together equal to two right angles, 15.

Art. 10. When two right lines meet another at the same point, but at different sides, and make angles with it which are together equal to two right angles, those right lines are in one continued right line, ibid.

Art. 11. If two right lines intersect one another, the vertically opposite angles are equal, 16.

Art. 12. If a right line intersect two parallel right lines, it makes the alternate angles equal to each other, 17.

Art. 13. If a right line intersect two parallel right lines, it makes the two internal angles on the same side of the intersecting line together equal to two right angles, 18.

Art. 14. If a right line intersect two parallel right lines, it makes each external angle equal to the farther internal angle on the same side of the intersecting line, ibid.

Art. 15. If a right line intersect two right lines, and make the alternate angles equal to each other, these two latter right lines are parallel, 19.

Art. 16. If a right line intersect two right lines, and make the external angle equal to the farther internal angle at the same side of the intersecting line, these two latter right lines are parallel, 20.

Art. 17. If a right line intersect two right lines, and make the two internal angles at the same side of the intersecting line together equal to two right angles, these two latter right lines are parallel, ibid.

Art. 18. If a right line intersect two parallel right lines, and another right line be drawn parallel to the intersector from any point in either of the parallels, it will meet the other, if produced sufficiently; and its length between the parallels will be equal to the length of the intersector between the parallels, ibid.

Art. 19. Right lines which join the adjacent extremities of two equal and parallel lines are themselves equal and parallel, 22.

Art. 20. The opposite sides of a parallelogram are equal, ibid.

Art. 21. The opposite angles of a parallelogram are equal, 23.

Art. 22. A parallelogram is divided into two equal parts by its diagonal, ibid.

Art. 23. Parallelograms on the same base, and between the same parallels, are equal, ibid.

Art. 24. Parallelograms on equal bases and between the same parallels are equal, 24.

Art. 25. Triangles upon the same base and between the same parallels are equal, ibid.

Art. 26. Triangles upon equal bases and between the same parallels are equal, ibid.

Art. 27. Equal triangles upon the same base, and upon the same side of it, are between the same parallels, 25.

Art. 28. Parallelograms which have equal bases and equal altitudes are equal, 26.

Art. 29. Triangles which have equal bases and equal altitudes, are equal, ibid.

Art. 30. Equal triangles on equal bases have equal altitudes, 27.

Art. 31. If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle, ibid.

Art. 32. All the sides of a square are equal, and all its angles right angles, 28.

Art. 33. Squares described upon equal right lines are equal, ibid.

Art. 34. If two squares be equal, their sides are equal, 29.

Art. 35. If any side of a triangle be produced, the external angle is equal to the two farther internal angles taken together, ibid.

Art. 36. The external angle of any triangle is greater than either of the two farther internal angles, ibid.

Art. 37. The three internal angles of any triangle taken together are equal to two right angles, ibid.

Art. 38. Any two angles of a triangle are together less than two right angles; and if any angle of a triangle be obtuse or right, the other two are acute: also, if two angles of a triangle be equal, they are both acute, ibid.

Art. 39. If two triangles have two angles in the one equal respectively to two angles in the other, the third angle of the one is also equal to the third angle of the other, 30.

Art. 40. The four internal angles of any four-sided rectilineal figure, taken together, are equal to four right angles, ibid.

Art. 41. In any triangle, if one side be greater than another, the angle opposite to that greater side is greater than the angle opposite to the lesser, ibid.

Art. 42. In any triangle, if one angle be greater than another, the side opposite to that greater side is greater than the side opposite to the lesser, ibid.

Art. 43. Any two sides of a triangle are together greater than the third side, 31.

Art. 44. If two triangles have two sides of the one equal respectively to two sides of the other, but the angle contained by each pair of these sides unequal—the base of that triangle whose given sides contain the greater angle is greater than the base of the other triangle, ibid.

Art. 45. If two triangles have two sides of the one equal respectively to two sides of the other, but their bases unequal,—the vertical angle of that triangle which has the greater base is greater than the vertical angle of the other triangle, 32.

Art. 46. If two triangles have two angles of the one equal respectively to two angles of the other, and a side of the one triangle equal to a corresponding side of the other—these triangles are in every respect equal to each other, ibid.

Art. 47. In any right-angled triangle the square described on the side

opposite the right angle is equal to the squares described on the sides containing the right angle, taken together, 33.

ART. 48. The centre of a circle falls within the circumference, 41.

ART. 49. A circle cannot have more than one centre, ibid.

ART. 50. A right line perpendicular to a chord through its middle point, will pass, if produced, through the centre of the circle, 42.

ART. 51. In a circle, a right line from the centre perpendicular to a chord divides it into two equal parts, 43.

ART. 52. In a circle, a right line, through the centre dividing a chord which does not pass through the centre into equal parts, is perpendicular to it, ibid.

ART. 53. A right line cannot meet the circumference of a circle in more than two points, ibid.

ART. 54. If a right line meet a circle in two points, that part of it between the points lies wholly within, and those parts of it not between the points lie wholly without the circle, 44.

ART. 55. In a circle, a perpendicular to a diameter, at its extremity, meets the circle in but one point, ibid.

ART. 56. If a right line be a tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent, 45.

ART. 57. If a right line be a tangent to a circle, the perpendicular to it at the point of contact will, if produced sufficiently, pass through the centre, 46.

ART. 58. The diameter of a circle is the greatest chord which can be drawn in it, ibid.

ART. 59. Chords equally distant from the centre of a circle are equal, 47.

ART. 60. Equal chords in a circle are equally distant from the centre, ibid.

ART. 61. In a circle the chord which is nearer to the centre is greater than that which is farther off, 48.

ART. 62. From any point which is not the centre of a circle, the greatest right line that can be drawn to the circumference is that which actually passes through the centre, ibid.

ART. 63. From any point which is not the centre of a circle, the least right line that can be drawn to the circumference is that which does not, but which would, if produced, pass through the centre, 49.

ART. 64. If from any point not the centre of a circle two right lines be drawn to the circumference, which make with that drawn through the centre, equal angles opening towards the same parts, these two lines are equal, ibid.

ART. 65. If from any point not the centre of a circle, but either within or on the circumference two right lines be drawn to the circumference, which make with the right line drawn actually through the centre, unequal angles opening towards the same parts, that which makes the smaller angle is greater than the other, 50.

ART. 66. If from a point outside a circle two right lines be drawn to the concave part of the circumference, which make with the right line through the centre unequal angles, that which makes the smaller angle is greater than the other, 51.

Art. 67. If from a point outside a circle two right lines be drawn to the convex part of the circumference, which make with the right line through the centre unequal angles, that which makes the smaller angle is less than the other, 52.

Art. 68. More than two equal right lines cannot be drawn to the circumference of a circle from any one point but the centre, ibid.

Art. 69. The opposite angles of a four-sided rectilineal figure inscribed in a circle are together equal to two right angles, 53.

Art. 70. The angles in the same segment of a circle are equal, 55.

Art. 71. The angle in the segment of a circle is half of the external angle at the centre whose sides are terminated in the extremities of the same segment of the circumference, ibid.

Art. 72. The angle in a semicircle is a right angle, 56.

Art. 73. The angle in a segment greater than the semicircle is less than a right angle, ibid.

Art. 74. The angle in a segment less than a semicircle is greater than a right angle, ibid.

Art. 75. If a tangent and a chord of a circle be drawn from the same point, the angle between them is equal to the angle in the alternate segment, 57.

Art. 76. If two different circles meet one another, they cannot have the same centre, 58.

Art. 77. One circle cannot meet another in more than two points, ibid.

Art. 78. If one circle meet another in two points, one portion of the former will be wholly within and the other wholly without the latter circle, ibid.

Art. 79. If two circles having their centres at the two extremities of a given finite right line pass through the same point on that finite line, they meet in that point, but in no other, ibid.

Art. 80. If two circles touch, the right line joining the centres, if produced, will pass through the point of contact, 59.

Art. 81. Equal circles have equal diameters, 60.

Art. 82. In equal circles equal chords cut off equal arches, ibid.

Art. 83. In equal circles equal arches have equal bases, ibid.

Art. 84. In equal circles, equal angles, whether they be at the centres or the circumferences, stand upon equal arches, 61.

Art. 85. In equal circles, the angles which stand upon equal arches are equal, whether they be at the centres or the circumferences, 62.

Art. 86. These latter four articles, it is evident, are true for the *same* circle as well as equal ones, 63.

Art. 87. In a circle parallel chords intercept equal arches, ibid.

Art. 88. In a circle, the chords joining the extremities of equal arches, and not intersecting, are parallel, 64.

Art. 89. If there be two right lines, one of which is divided into any number of parts, the rectangle under the two lines is equal to the sum of the rectangles under the undivided line and the several parts of the divided line, 65.

Art. 90. If a right line be divided into any two parts, the square of the whole line is equal to the sum of the rectangles under the whole line and each of the parts, ibid.

Art. 91. If a right line be divided into any two parts, the rectangle under the whole line and either part is equal to the square of this part together with the rectangle under the parts themselves, 66.

Art. 92. If a right line be divided into any two parts, the square of the whole line is equal to the sum of the squares of the parts together with twice the rectangle under the parts, ibid.

Art. 93. The square of a right line is equal to four times the square of its half, 67.

Art. 94. Parallelograms which have equal altitudes, have to each other the same ratio as their bases, 75.

Art. 95. Parallelograms which have equal bases, have to each other the same ratio as their altitudes, 76.

Art. 96. Two equal parallelograms, which are also equi-angular, have the sides about their equal angles reciprocally proportional, 77.

Art. 97. Two equi-angular parallelograms which have their sides about their equal angles reciprocally proportional are equal, 78.

Art. 98. Equal parallelograms have their bases and altitudes reciprocally proportional, ibid.

Art. 99. Parallelograms which have their bases and altitudes reciprocally proportional are equal, 79.

Art. 100. If four right lines be proportionals, the rectangle under the extremes is equal to the rectangle under the means, ibid.

Art. 101. If there be four right lines, and the rectangle under any two of them equal to the rectangle under the remaining ones, these right lines are four proportionals, ibid.

Art. 102. If three right lines be proportionals, the rectangle under the extremes is equal to the square of the mean, 80.

Art. 103. If there be three right lines, and the rectangle under any two of them equal to the square of the third, these three right lines are proportionals, ibid.

Art. 104. Triangles which have equal altitudes are to each other as their bases, 81.

Art. 105. Triangles which have equal bases are to each other as their altitudes, ibid.

Art. 106. Equal triangles which have also an angle in the one equal to an angle in the other, have the sides about these equal angles reciprocally proportional, ibid.

Art. 107. Two triangles which have an angle in one equal to an angle in the other, and have also the sides about these equal angles reciprocally proportional, are equal, 82.

Art. 108. Equal triangles have their bases and altitudes reciprocally proportional, ibid.

Art. 109. Triangles which have their bases and altitudes reciprocally proportional are equal, ibid.

Art. 110. If a right line be drawn parallel to any side of a triangle, and meeting the other sides, the segments of one of these sides have the same ratio as the segments of the other, 83.

Art. 111. If a right line meets the sides of a triangle, so that the segments of one side shall have the same ratio as the segments of the other,—and if the corresponding segments be at the same side of the

intersecter,—this right line is parallel to the remaining side of the triangle, 83.

Art. 112. If the angles of one triangle be respectively equal to the angles of another, the three sides of one triangle have to the corresponding sides of the other, respectively, the same ratio, ibid.

Art. 113. If the three sides of any triangle have respectively to the three sides of another the same ratio, the angles of one triangle are respectively equal to the angles of the other, 87.

Art. 114. If two sides of any triangle have respectively to two sides of another the same ratio, and likewise the angles contained by each pair of sides equal,—the other angles of the triangles will be also respectively equal, ibid.

Art. 115. If two triangles have an angle in the one equal to an angle in the other; and if the sides containing a second angle in the former have, respectively, the same ratio to the sides containing a second angle in the latter; and if likewise the third angles of the triangles are either both acute or both obtuse, or both right; all the angles of these triangles are respectively equal to each other, 88.

Art. 116. Equi-angular parallelograms have to each other the ratio compounded of the ratios of the sides about the equal angles, 91.

Art. 117. Triangles which have an angle in the one equal to an angle in the other, have to one another a ratio compounded of the ratios of the sides about the equal angles, ibid.

Art. 118. If the angles of one triangle be respectively equal to those of another, these triangles have to each other a ratio duplicate of that which their corresponding sides have to each other, 92.

Art. 119. In equal circles, angles at the centres have to each other the same ratio as the arches on which they stand, 93.

Art. 120. In equal circles, angles at the circumferences have to each other the same ratio as the arches on which they stand, 94.

Art. 121. Every triangle which has its three sides equal has also its three angles equal, 95.

Art. 122. Every triangle which has its three angles equal has also its three sides equal, ibid.

Art. 123. Where several right lines meeting another right line at the same point, make angles with it, these angles are altogether equal to two right angles, ibid.

Art. 124. Two right lines intersecting each other, make angles which taken together are equal to four right angles, ibid.

Art. 125. If several right lines intersect one another in the same point, all the angles taken together are equal to four right angles, 96.

Art. 126. A triangle which has two of its sides equal, if these equal sides be produced, will have the angles beneath the third side equal to each other, ibid.

Art. 127. If two right lines be parallel to the same right line, they are parallel to one another, 98.

Art. 128. If a right line intersect two right lines, and make the two internal angles at the same side of the intersecting line together less than two right angles, these two latter right lines will meet if produced sufficiently, 99.

Art. 129. If a parallelogram and triangle be upon equal bases and between the same parallels, the parallelogram is double of the triangle, 100.

Art. 130. If a parallelogram and triangle be between the same parallels, and the base of the triangle double the base of the parallelogram, then the parallelogram and triangle are equal, ibid.

Art. 131. In a given parallelogram the complements of the parallelograms about the diagonal are equal to each other, 101.

Art. 132. If the square described upon one side of a triangle be equal to the squares described on the other sides of the triangle, taken together, the angle opposite to the first-mentioned side is a right angle, ibid.

Art. 133. Two perpendiculars cannot be drawn from the same point to the same right line, 102.

Art. 134. If two right lines be drawn from the same point to the same given right line, and if one of them be perpendicular, the other not, the perpendicular will fall at the side of the acute angle, ibid.

Art. 135. If two right lines be drawn from the same point to the same right line, and if one of them be perpendicular, the other not, the perpendicular is less than the other, ibid.

Art. 136. Each angle of an equal-sided triangle is one-third of two right angles, 103.

Art. 137. In a right-angled triangle whose sides about the right angle are equal, the remaining angles are each equal to half a right angle, ibid.

Art. 138. If from a point within a triangle right lines be drawn to the extremities of any side, these are together less than the other two sides of the triangle, but contain a greater angle, ibid.

Art. 139. In a triangle which has two equal sides, the right line drawn from the vertex of the angle between them, perpendicular to the third side, divides that angle, and also the third side into two equal parts, respectively, 104.

Art. 140. In a triangle which has two equal sides, a right line dividing the angle between them into two equal parts, if drawn to the third side, will divide it into two equal parts, and also be perpendicular to it, ibid.

Art. 141. In a triangle which has two equal sides, a right line drawn from the vertex of the angle between them to the middle point of the third side, divides the opposite angles into two equal parts, and is also perpendicular to the third side, ibid.

Art. 142. In a circle, if two chords intersect, which are not both diameters, they do not divide each other into equal parts, 105.

Art. 143. If from a point within a circle there can be drawn more than two equal right lines to the circumference, this point is the centre of the circle, ibid.

Art. 144. If a right line be drawn from the point of contact nearer the centre than the tangent, it cuts the circle, 106.

Art. 145. At the same point of the same circle but one right line can be drawn touching the circle, ibid.

Art. 146. If one side of a quadrilateral figure inscribed in a circle be produced, the external angle thus formed is equal to the internal remote angle of the quadrilateral, ibid.

Art. 147. The difference between the squares of any two unequal

right lines is equal to the rectangle under the sum of the lines and their difference, 107.

Art. 148. If a right line be equally divided, and produced to any point,—then, the rectangle under the whole line and the produced part, is equal to the difference between the square of half the original line, and the square of the line made up of that half and the produced part, ibid.

Art. 149. If a right line be divided into two equal parts, and into two unequal parts, the rectangle under the unequal parts is equal to the difference between the square of half the line, and the square of the intermediate part, 108.

Art. 150. If a right line be divided equally and unequally, the rectangle under the equal parts is greater than the rectangle under the unequal parts, ibid.

Art. 151. If a right line be cut equally and unequally, the sum of the squares of the unequal parts is greater than the sum of the squares of the equal parts,—or, in other words, greater than twice the square of half the line, ibid.

Art. 152. If two equal triangles have an angle in the one which together with the angle in the other is equal to two right angles, the sides about these angles are reciprocally proportioned, 109.

Art. 153. If two triangles have an angle in the one which together with the angle in the other is equal to two right angles, and if the sides about the given angles are reciprocally proportional, then these two triangles are equal, ibid.

Art. 154. If four right lines be proportionals, the parallelogram under the extremes is equal to an equi-angular parallelogram under the means, ibid.

Art. 155. If two chords of a circle intersect, the rectangle under the segments of one is equal to the rectangle under the segments of the other, 110.

Art. 156. If from a point without a circle a secant and a tangent be drawn to the circle, the square of the tangent is equal to the rectangle under the whole secant and its external part, ibid.

Art. 157. If from a point without a circle two right lines be drawn, one cutting the circle, the other meeting it at any point, and if the rectangle under the whole secant and its exterior part be equal to the square of the line which meets the circle,—then, this line is a tangent, ibid.

Art. 158. A right line parallel to any side of a triangle, and meeting the other sides produced, cuts them so that the segments of the one have the same ratio as the segments of the other, 111.

Art. 159. A right line cutting any two produced sides of a triangle, so as to make the segments of the one proportional to the corresponding segments of the other, is parallel to the third side, 112.

Art. 160. If a right line parallel to any side of a triangle divides either of the other sides equally, it will divide both equally, ibid.

Art. 161. If a right line divide two sides of a triangle equally, it is parallel to the third, ibid.

Art. 162. If several right lines be drawn parallel to a side of any triangle, and meeting the other sides, the segments of one of these sides have the same ratio as the corresponding segments of the other, ibid.

Art. 163. A right line dividing any angle of a triangle into two equal parts, divides the opposite sides into segments which have the same ratio as the sides which contain the divided angle, 113.

Art. 164. If a right line drawn from any angle of a triangle to the opposite side divide that side into parts, which have the same ratio as the corresponding sides about the given angle, this angle is divided into two equal parts, 114.

Art. 165. In a right-angled triangle, a perpendicular drawn from the vertex of the right angle to the opposite side, is a mean proportional between the segments of this side, ibid.

Art. 166. In such a triangle as above described, each side about the right angle is a mean proportional between the corresponding segment and the side opposite the right angle, 115.

Art. 167. In such a triangle as above described, the three sides and the perpendicular are proportionals, ibid.

Art. 168. In the same circle, the angles, whether at the centre or circumference, have the same ratio to each other as the arches they stand on, ibid.

Art. 169. In a circle, any angle is to four right angles as the arch on which it stands to the whole circumference, ibid.

Art. 170. In a series of four proportional right lines, the second is to the first as the fourth to the third, 116.

Art. 171. In a series of four proportional right lines, the first is to the third as the second to the fourth, ibid.

Art. 172. In a series of four proportional right lines, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth, ibid.

Art. 173. In a series of four proportional right lines, the difference between the first and second is to the second as the difference between the third and fourth is to the fourth, 117.

Art. 174. In a series of four proportional right lines, the first is to the sum of the first and second as the third is to the sum of the third and fourth, ibid.

Art. 175. In a series of four proportional right lines, the first is to the difference between the first and the second as the third is to the difference between the third and the fourth, 118.

A

POPULAR SYSTEM OF GEOMETRY.

A SPACE may have length, breadth, and depth; or length and breadth only; or length alone. These are called its *dimensions*.

SOLIDS are magnitudes which have the *three* dimensions of space: length, breadth, and depth. Thus a *die*, or any other magnitude having these three dimensions, is a solid.

SURFACES are magnitudes which have but *two* dimensions: length and breadth. Thus the *face* of a die is a surface; having length and breadth, but not depth,—else it would be, not the face of the die, but part of the solid.

LINES are magnitudes which have but *one* dimension: length. Thus the *edge* of a die is a line; having length, but not breadth,—else it would be, not the edge of the die, but part of a surface.

A *Right* Line is a line which is *perfectly even* or *straight* throughout its whole extent*.

A *Plane* Surface is a surface which is *perfectly even* or *flat* throughout its whole extent.

A *Mathematical* Solid is a solid with its surface or surfaces either plane, or, if not, *perfectly smooth*, throughout the whole extent of each particular surface.

GEOMETRY is that Science which treats of Right Lines, Plane Surfaces, and Mathematical Solids, with regard to magnitude.

PLANE GEOMETRY is that Science which treats of the Right Line and Circle. [See NOTE A.]

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* A *right* line and a *straight* line are, in geometry, synonymous terms, but it is better to use the former, which always indicates a *perfectly* straight line, whilst the word "straight" is often applied in common speech to lines which are not perfectly straight, but only nearly and *visibly* so.

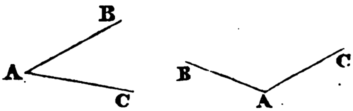
THE ELEMENTS OF GEOMETRY.

PART I.

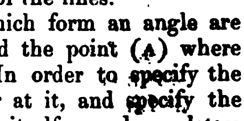
Of Right Lines and Rectilineal Figures.

DEFINITION I. A rectilineal (or right-lined) *Angle* is that which is formed by two right lines meeting at the same point, but not lying in the same right line.

Thus in the annexed plate, the right lines A B and A C which meet at the point A, but are not in the same right line, form a rectilineal angle.



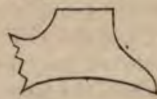
It is to be observed that by an angle is not meant the *surface* between the lines which form it; for though we increased that surface by producing the lines (to D and E, for instance), the angle will still remain the same in magnitude. By an angle in fact is meant the *degree of increasing width*, or *separation*, between the lines which form it; this depends, not upon the *length*, but upon the *direction* of the lines.



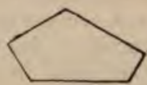
The right lines (as AD, AE) which form an angle are usually called its *sides* or *legs;* and the point (A) where they meet is called its *vertex*. In order to specify the vertex of an angle we place a *letter* at it, and specify the letter: in order to specify the angle itself, we place a letter at each side of it and another at the vertex; and specify the two former letters with the other *between* them. Thus, in the above plate, we specify the vertex by mentioning the letter A; and we specify the angle itself by mentioning the letters BAC or CAB.

DEF. II. *Figures* are bounded portions of space.

DEF. III. A *plane* figure is a plane surface bounded by one or more lines.



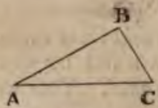
DEF. IV. A plane *rectilineal* figure is a plane surface bounded by *right* lines.



LESSON I.

DEF. V. A rectilineal triangle is a plane figure bounded by *three* right lines.

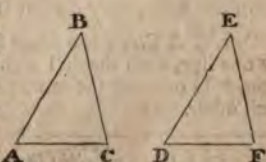
It is so called because the three lines form with each other three *angles*. A triangle is specified by mentioning, in any order, three letters placed respectively at the vertices of its three angles. Thus the above triangle is called the triangle ABC.



If we conceive a triangle to stand upon any one of itssides, that one is called the *base* of the triangle, and the other two are called its legs or sides. Also the angle opposite to the base is called the *vertical* angle; and *its* vertex the vertex of the triangle. Thus, in the above figure, if we conceive the triangle to stand upon the side AC, this side is called the base; the angle at B is called the vertical angle; and the point B the vertex of the triangle. [See NOTE B.]

ARTICLE 1. *If there be two triangles which have two sides of the one equal, respectively, to two sides of the other; and likewise the angles contained by those sides equal to one another:—then, the bases or third sides of these triangles are also equal.*

Let ABC, DEF be two triangles, having the side BA equal to the side ED, and the side BC equal to the side EF, also the angle ABC equal to the angle DEF. Then, also, the base AC must be equal to the base DF.



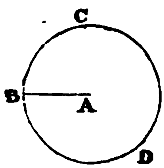
DEMONSTRATION. Conceive the triangle ABC so applied to the triangle DEF, that the point B may be on the point E, and the side BA upon the side ED; moreover, that the side BC may lie towards the *same hand* as the side EF. Then, the point B falling on the point E, and the side BA on the side ED, the point A would necessarily fall on the point D, because the *sides* BA, ED are equal. Likewise, the side BC would fall upon the side EF, because the angles ABC, DEF are equal in width. The point C would likewise fall on the point F, because the point B falls on the point E, and the sides BC, EF are equal. Therefore, as it has been shewn that the points A and C would coincide with the points D and F respectively, the right lines AC and DF would coincide exactly throughout their whole extent, else they would enclose a space between them, which two *right* lines cannot do, as is manifest *. Hence, inasmuch as AC and DF would coincide exactly, they must be exactly *equal*. This was the assertion of the present Article.

PROBLEMS.

DEF. 1. An *equilateral* triangle is a triangle whose three sides are equal to each other.

Observation. One extremity of a definite right line remaining fixed, if the line be made to revolve about this point, it is evident that the other extremity will trace out a line which is every where equally distant from the fixed point: and that if the line revolve progressively to its first direction, the line traced out will return into itself, so as totally to include a surface.

Thus, if AB be the right line, A its fixed extremity, the line BCDB will be traced out by the progressive revolution of AB round the point A, through the points C, D, to its first direction AB. Also every point of this line BCDB will be equally distant from A.



DEF. 2. A *Circle* is a plane figure bounded by one line, such that all right lines drawn from it to one and the same point within the circle, are equal to each other.

A xiom 1. *Two right lines cannot completely enclose a space.*

ART. 2. *In such triangles as above described, those angles at the bases which are opposite to equal sides, are respectively equal.*

In the above figures the angles at A and D, opposite the equal sides BC and EF, are equal; also the angles at C and F, opposite the equal sides BA, ED, are equal.

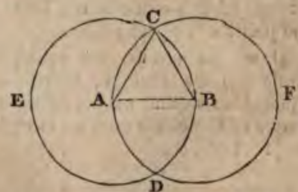
DEM. In the preceding demonstration it was shown that the sides of the angle BAC would coincide exactly with the sides of the angle EDF; also that the sides of the angle BCA would coincide exactly with the sides of the angle EFD. Hence, the angles BAC, EDF are necessarily equal; and also the angles BCA, EFD*. This was the assertion, &c.

ART. 3. *Such triangles as above described are equal, in every respect, to each other.*

For in the same demonstration it was shown that the three sides of the triangle ABC would coincide exactly with the three sides of the triangle DEF. Hence these triangles must be in every respect equal*. This was, &c. [See NOTE C.]

DEF. 3. In a circle, the bounding line is called the *Circumference*, and the point from which it is every where equally distant is called the *Centre* of the circle.

PROB. I. *To describe an equilateral triangle on a given finite right line.*



Let AB be the given finite right line. It is required to describe an equilateral triangle upon it.

CONSTRUCTION. With the point A as a centre, and the line AB as distance, describe the circle DBE. Also, with the point B as a centre and the line BA as distance, describe the circle DAF. Draw the right lines AC, BC, from the points A and B respectively to either point, as C,

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* AX. 2. *Magnitudes which coincide exactly with each other are equal.*

This theorem, which is made the fourth proposition in Euclid, should rightly be the first, inasmuch as it is the foundation of Geometry, the corner-stone upon which the whole superstructure of geometrical science rests. Yet viewing it without fear or prejudice, we see how nearly

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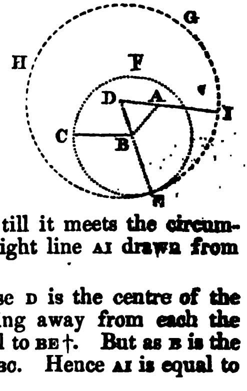
where these circles intersect. Then, the triangle ACB is equilateral.

DEMONSTRATION. As DBE is a circle, of which A is the centre, AC is equal to AB, by DEF. 3; and as DAF is a circle, of which B is the centre, BC is equal to BA, by the same definition. Hence AC and BC, being both equal to AB, are equal to each other, and the three sides of the triangle are equal *. This was what was required to be done by the present Article.

PROB. II. *From a given point to draw a right line equal to a given finite right line.*

Let A be the given point, and BC the given right line. It is required to draw from the point A a right line equal to BC.

CONS. Draw the straight line AB from the given point to either extremity of the given line, and upon AB describe the equilateral triangle ADB, by PROB. I. With B as a centre and BC as distance describe the circle CEF, and produce DB through the point B till it meets the circumference at E. With the point D as a centre, and the distance DE, describe the circle EGH, and produce DA through the point A till it meets the circumference of this circle at I. Then the right line AI drawn from the point A is equal BC.



DEM. DI and DE are equal, because D is the centre of the circle EIG, by DEF. 3; therefore, taking away from each the equal sides DA and DB, AI remains equal to BE †. But as B is the centre of the circle ECF, BE is equal to BC. Hence AI is equal to BC ‡. This, &c.

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* AX. 3. *Things which are equal to the same thing are equal to each other.*

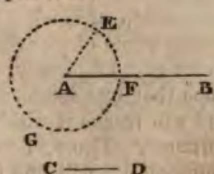
† AX. 4. *If from equal things we take away equal things, respectively, the remainders will be equal.*

‡ By AX. 3.

connected with our simplest thoughts and habits, Geometry, the most abstract and refined of all human inventions, really is. The demonstration in Article the first is little more than shewing that the *palms of our hands* are equal by putting them together! If a carpenter has two triangular pieces of wood, and desires to know whether they are exactly equal, what does he? Why, he applies one to the other, in the manner we suppose above, and thereby forms his opinion. Such, and so little removed from common daily practice, is our demonstration. It is of great use that the reader should consider the principles of Geometry in this familiar way; he will perceive by this means that there is truly nothing so very abstruse or difficult in the Science,—nothing to affright him with the thoughts of attempting and mastering it. In short, the demonstration of the present theorem, the groundwork of the whole science, differs from a mere mechanical application of two pieces of paper, shaped in a certain triangular manner, in nothing whatever but this, namely: instead of actually applying one triangle to the other, we only *suppose* one triangle to be applied to the other. By this subterfuge it is that geometers have contrived to give an air of abstractness and purity to their science; but it is really founded on the very simple and practical basis of admeasurement to which we have alluded.

PROB. III. *From the greater of two given right lines to cut off a part equal to the less.*

Let AB, CD be two right lines, of which AB is greater than CD. It is required to cut off from AB a part equal to CD.



CONS. From either extremity of the greater line A draw AE equal to CD by preceding PROB.; and with the point A as a centre, and the distance AE, describe the circle EFG, cutting AB in F. Then AF is equal to CD.

DEM. By DEF. 3, AF is equal to AE. Hence AF is also equal to CD*. This, &c.

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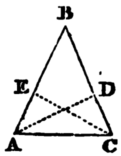
* By Ax. 3.

ART. 4. *A triangle which has two of its sides equal, has also its angles, opposite the equal sides, equal.*

In the triangle ABC let the side BA be equal to the side BC. Then, also, the angle BCA is equal to the angle BAC.



DEM. Let D be any point on the side BC, and take E a point equally distant from the vertex B, on the side BA*; let DA, EC be right lines joining these points with the vertices of the opposite angles. Therefore, in the two triangles BAD, BCE, there are two sides BA, BD of the one, equal respectively to two sides BC, BE of the other; likewise, the angle contained by the former pair of sides is equal to the angle contained by the latter, for it is the same angle ABC. Consequently, by ART. 2, the angle BAD is equal to the angle BCE. In the same manner, if on the sides BA, BC, there be any other two points *equally distant* from the vertex B; and if these points be joined as above with the vertices of the angles, C and A, opposite them respectively; it can be shown as above that the angles contained between these joining lines and the sides are equal. But A and C are two such points, being on the sides, and equally distant from the vertex B by the terms of the Article:—hence, if *these* points A and C be joined respectively with the vertices C and A, as in this case the joining lines would fall upon AC, the angles between them and the sides are BCA and BAC, which, in the same manner as before, are equal. This was the assertion, &c.

This theorem, which nearly corresponds with the fifth proposition of Euclid, is known at schools and universities by the name of *pons asinorum*, or "asses' bridge," on account of its difficulty, as given in Euclid's Elements. We hope that our demonstration renders it sufficiently simple, and will rescue it with our readers from such a disgraceful surname. Those, however, who may prefer Euclid's demonstration, will find it in the notes. [See NOTE D.]

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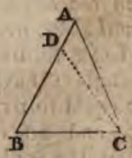
* If the reader wishes to see how this may be actually done, let him read PROBLEMS I., II., and III.

ART. 5. *A triangle which has two of its angles equal, has also its sides opposite the equal angles, equal.*



In the triangle BAC, let the angle ABC be equal to the angle ACB. Then, also, the side AC is equal to the side AB.

DEM. For *suppose* BA greater than AC; and that a portion of BA, such as BD, were equal to AC: then, if the right line DC were drawn, we should have two triangles ACB, DBC, with the sides AC, CB of the one equal respectively to the sides DB, BC of the other; also, by the terms of this ART. the angle ACB contained by the two former, is equal to the angle DBC contained by the latter. Consequently, by ART. 3, the triangles ACB, DBC would be equal—*on the above supposition*. But these triangles being evidently *not* equal*, the above supposition must be false; that is, AB cannot be greater than AC. In the same manner it may be shown that AC is not greater than AB. Hence, as neither of the sides AB, AC is greater than the other, they must be equal. This, &c.

This is what mathematicians call an indirect proof, or an argument *ad absurdum*. In it, by *supposing* the contrary of our assertion to be true, and thence legitimately deducing a *false* conclusion, we in fact prove that this supposed contrary is *not* true,—or, in other words, that our assertion is true. Thus, by *supposing* BA unequal to AC, we come to the conclusion that two triangles are equal, which are evidently not so; therefore BA is not unequal to AC, that is, it is equal to it.

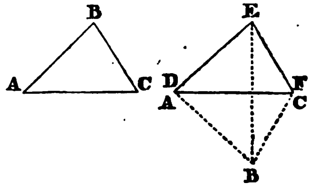
In the common geometries, the hypothetical manner of speaking is lost sight of during the demonstration; whereby the learner is apt to forget its indirect nature. We have carefully attended to this circumstance, and carried on the suppositional phraseology throughout, till we reach the absurd conclusion. [See NOTE E.]

ART. 6. *If there be two triangles which have all the sides of the one equal, respectively, to all the sides of the other,—*

* Ax. 5. *The whole is greater than its part.*

then the angles, which in these triangles are opposite to equal sides, are also equal.

Let ABC, DEF be the two triangles; having the side AC equal to DF, AB equal to DE, and BC equal to EF. Then, the angle at B is equal to that at E, the angle at C equal to that at F, and the angle at A equal to that at D.

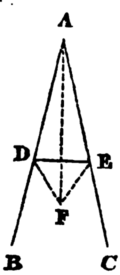


DEM. Conceive the triangle ABC so applied to the triangle DEF that the point A may fall on the point D, and the side AC on the side DF; moreover that B may lie at a *different side* of DF from E. Then, the point A falling on the point D, and the side AC on the side DF, the point C would necessarily fall on the point F, because the sides AC, DF are equal. If, therefore, the right line EB were drawn, there would be formed two triangles EDB, EFB, the one having its sides DE, DB equal, the other having its sides FE, FB equal, by the terms of this Article. Consequently, by Art. 4, the angle DEB would be equal to the angle DBE, and the angle FEB

PROB. IV. *To divide a given rectilineal angle into two equal parts.*

Let BAC be the given angle. It is required to divide it into two equal parts.

CONS. On the sides AB and AC respectively take the points D and E equally distant from A. Join DE, and upon DE describe the equilateral triangle DFE by PROB. I., so that it may fall at a different side of DE from A. Then the right line AF will divide the angle BAC into two equal parts.



DEM. In the triangles FAD, FAE, the line AD is equal to AE, DF to EF, and the side AF common. Hence, by ART. 6, the angle FAD is equal to the angle FAE. This, &c.

The angle may be divided into four equal parts, by again dividing FAD and FAE into two equal parts by this problem. In like manner it may be divided into 8 equal parts, or 16, or 32, &c.

to the angle FBE. Hence, the angle DEF must be equal to the angle DBF, that is, to the angle ABC*.

Again: As the two triangles ABC, DEF, have two sides AB, BC of the one equal respectively to two sides DE, EF of the other, and also, as we have just shown, the angles contained by those sides, ABC, DEF, equal; the remaining angles opposite equal sides are, by ART. 2, equal; that is, the angle at C to the angle at F, and the angle at A to that at D. This, &c. [See NOTE F.]

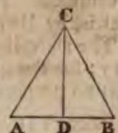
ART. 7. *Such triangles as are described in the preceding Article are equal, in every respect, to each other.*

By ART. 3: because the sides BA, BC are respectively equal

PROB. V. *To divide a given finite right line into two equal parts.*

Let AB be the right line. It is required to divide it into two equal parts.

CONS. Upon AB describe the equilateral triangle ACB, by PROB. I.; and by PROB. IV. divide the angle ACB into two equal parts by the line CD. Then if CD be produced to meet AB, it will divide AB into two equal parts.

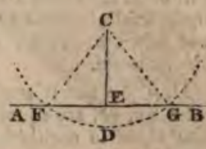


DEM. In the triangles ACD, BCD, the sides CA and CB are equal, the side CD is common, and the angle ACD is equal to BCD. Hence, by ART. I, AD is equal to DB. This, &c.

PROB. VI. *To draw a perpendicular to a given right line, from a given point without it.*

Let AB be the given line, C the point without it. It is required to draw from the point C a perpendicular to AB.

CONS. Take any point, D, at the other side of AB from C, and with C as a centre and CD as distance, describe the circle FDG, cutting AB in the points F and G. Divide FG into two equal parts at E, by preceding PROB., and draw CE. Then CE is perpendicular to AB.



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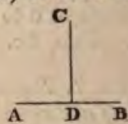
* AX. 6. *If to equal things we add equal things, respectively, the wholes will be equal.*

to the sides ED, EF, and the angle ABC is equal to the angle DEF.

LESSON II.

DEF. VI. If one right line, standing upon another, make the adjacent angles equal to one another, each of these angles is called a *right* angle, and the right line which stands upon the other is called a *perpendicular* to it.



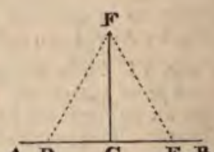
Thus if the right line CD stand upon the right line AB, and make the angles CDA, CDB equal to one another, each of these angles is called a right angle; and the line CD is called a perpendicular to the line AB. [See NOTE G.]

DEM. In the triangles FCE, GCE, the side CF is equal to CG, by DEF. 3.; and the side EF to EG, by construction: and the side CE is common. Hence, by ART. 6, the angles CEF, CEG are equal; that is, by DEF. VI., CE is perpendicular to AB. This, &c.

The right line is supposed capable of being produced if necessary; for the given point might be so situated with respect to it that no perpendicular could be drawn to it unless produced.

PROB. VII. *To draw a perpendicular to a given right line, from a point in it.*

Let AB be the given line, C the point in it. It is required to draw from C a perpendicular to AB.



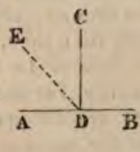
CONS. At different sides of C take CD, CE, equal to each other, by PROB. III. Upon DE describe the equilateral triangle DFE, by PROB. I. and join FC. Then, FC is perpendicular to AB.

DEM. In the triangles DFC, EFC, the side FC is common, the sides DF, FE, are equal to each other, and the sides DC, CE, are also equal to each other. Hence, by ART. 6, the angle FCD is equal to the angle FCE; that is, FC is perpendicular to AB, by DEF. VI.

The right line in this problem also is supposed capable of being produced if necessary; as the given point might be at one of its extremities.

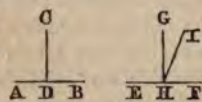
DEF. VII. An angle greater than a right angle is called an *obtuse* angle; and an angle less than a right angle is called an *acute* angle.

Thus, if CD be perpendicular to AB, the right line DE from the point D makes with AB two angles; one of which EDB, being greater than the right angle CDA, is called obtuse, and the other, EDA, being less than the right angle CDA, is called acute.



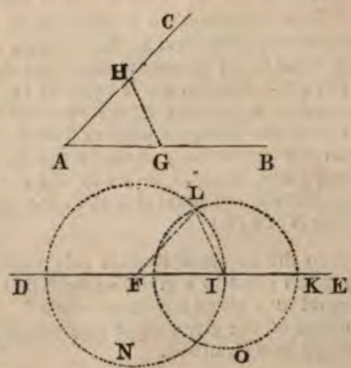
ART. 8. *All right Angles are equal.*

Let CDB, GHE be any two right angles. Then the angle CDB is equal to the angle GHE.



DEM. Produce BD through D to any point A, and since CDB is granted a right angle, the line CD must form with the line AB the two angles CDA, CDB, equal, by DEF. VI.

PROB. VIII. *At a given point in a given indefinite right line, to draw a right line making, with the given one, an angle equal to a given angle.*



Let BAC be the given angle, DE the given line, and F the given point in it. It is required to make an angle equal to BAC at the point F, one side of which shall be the right line FE.

CONS. On AB take any point G, and from AC cut off AH equal to AG, by PROB. III.; join HG. Likewise, from the line FE cut off FI equal to AG, by PROB. III.; and with F as a centre, describe the circle NLI, at the distance FI. From the line IE cut off IK equal to GH, by PROB. III.; and with I as a centre, describe the circle

For the same reason, producing EH through H to any point F, the angles GHE, GHF are equal. Now conceive the line AB so applied to the line EF, that the point D may fall on the point H, and the line AB may fall along the line EF; also that DC may be at the same side of EF with HG: then, DC would necessarily fall along HG. For *suppose* it to fall in any other direction, as HI; consequently, the angles GHE, GHF, being equal, the angles IHE, IHF are unequal,—that is, the angles CDA, CDB (which are the same as IHE, IHF) would be, *on this supposition*, unequal. But they have been proved equal; therefore this supposition is false, the line DC would *not* fall in any other direction but HG. Hence the angles CDB, GHE must be equal, inasmuch as the point D being on the point H, and the line AB on the line EF, the side DC would fall on the side HG*. This &c.

This principle is omitted in Euclid; but is necessary to a rigorous system of demonstration, such as the Elements of Geometry should exhibit.

OLK, at the distance IK. Join the points F and I with the point L, where these circles intersect. Then the angle IFL is equal to the angle BAC.

DEM. Because AG is equal to FI by construction, and FI is equal to FL, by DEF. 3; therefore AG and FL are equal†. But AG is equal to AH by construction; and therefore AH is likewise equal to FL †. Finally, GH is equal to IK, by construction, and IK is equal to IL, by DEF. 3: therefore GH and IL are equal†. Hence, in the two triangles GAH, IFL, since the three sides AG, AH, and GH, are respectively equal to the three sides FI, FL, and IL, the angle GAH opposite to GH in the one, is equal to the angle IFL opposite to the equal side IL in the other, by ART. 6. This, &c. [See NOTE H.]

In the common Euclids this problem is enounced thus: "At a given point in a given straight line to make a rectilineal angle equal to a given rectilineal angle." This is imperfect, the condition of the given right line forming *one side* of the required angle being omitted.

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* By Ax. 2. † By Ax. 3.

ART. 9. *When a right line meeting another right line makes angles with it, these angles are together equal to two right angles.*

Fig. 1.

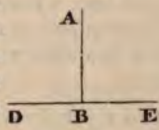
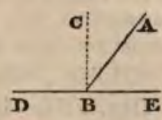


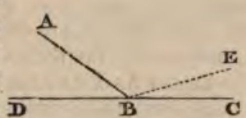
Fig. 2.



Let the right line AB meet the right line DE, and make angles with it ABD, ABE. Then, the angles ABD, ABE taken together are equal to two right angles.

DEM. If, as in fig. 1, AB be perpendicular to DE, the angles ABD, ABE are both right angles by DEF. 6. Hence both together are equal to two right angles. If, as in fig. 2, AB should not be perpendicular to DE, the angles ABD, ABE, taken together, are evidently equal to the angles CBD, CBE made by the perpendicular CB*, taken together. Hence the angles ABD, ABE, taken together, are equal to two right angles, inasmuch as the angles CBD, CBE, by the preceding part of the demonstration are equal to two right angles†. This, &c.

ART. 10. *When two right lines meet another, at the same point, but at different sides, and make angles with it which are together equal to two right angles, those right lines are in one continued right line.*



Let DB and CB be the two lines meeting AB at the point B, but at opposite sides of AB; and forming the angles ABD, ABC, which are together equal to two right angles. Then DB and BC are in one and the same right line.

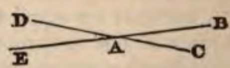
DEM. For, let it be *supposed* that BC is not in the same continued right line with DB, but that any other right line BE is the continuation of DB. On this supposition the angles ABD, ABE taken together would be equal to two right angles, by preceding ART.; but the angles ABD, ABC taken together are granted equal to two right angles;

* To see how a perpendicular is to be raised from a given point in a given line, read PROBLEMS down to number VII.

† By Ax. 3.

therefore* ABD, ABE together would be equal to ABD, ABC together by ART. 8. Consequently, if from these equal quantities we take away, respectively, the angle ABD which is *common to both*, the remaining angle ABE would be equal to the remaining angle ABC, a *part* to the *whole*, which is impossible. The above supposition, therefore, was false; BE is not in a right line with BD, nor is any other line except BC. This, &c.

DEF. VIII. If two right lines cut each other so as to form *four* angles, each opposite pair are called *Vertically opposite* angles.



Thus the angles BAC, DAE are vertically opposite angles; as also DAB, EAC. The lines DC, EB, are said to *intersect*.

ART. 11. *If two right lines intersect one another, the vertically opposite angles are equal.*

Let DC, EB intersect one another, as in the preceding figure. Then the angles BAD, CAE are equal; as also the angles DAE, BAC.

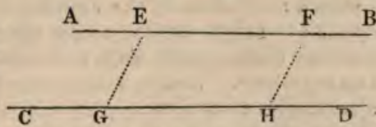
DEM. By ART. 9, as the right line BA meets the right line DC, and makes angles with it, these angles BAC, BAD, are together equal to two right angles. For a similar reason the angles BAC, CAE, made by the right line CA with the right line EB, are together equal to two right angles. Consequently the angles BAC, BAD together are equal to the angles BAC, CAE together*. Hence, from these equals taking away the common angle BAC, the angle BAD remains equal to the angle CAE†. In the same manner it can be proved that the angles BAC, DAE, are equal. This, &c.

LESSON III.

DEF. IX. Two right lines are said to be equally distant from one another when any two points whatsoever in the one not the greater, and any two equally remote points in the other, being taken, the right lines which join each

* By Ax. 3. † By Ax. 4.

opposite pair of points towards the same hand are equal to each other.



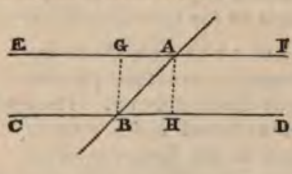
Thus, let AB, CD, be the two right lines; E, F, any two points in AB which is equal to, or less than the whole line CD; and G, H, any two points equally remote in the line CD. Then, if the right lines EG, FH, be equal, every where such points can be taken, the lines AB, CD, are said to be equally distant from each other. [See NOTE I.]

DEF. X. *Parallel* right lines are those which are equally distant from each other.

This is the simple, clear, and familiar notion of parallels. Thus, if we wish to know whether two straight *walls* or *trenches* are parallel to each other, we measure the interval between them at two different points, by taking two equally remote points (generally those directly facing the first ones) in the opposite wall or trench: if the intervals measured be equal, we conclude the walls or trenches are parallel.

Obs. It is evident from our definition that parallel right lines never meet; and that the definition of parallels equally applies to lines of a finite or an infinite length. [See NOTE J.]

ART. 12. *If a right line intersect two parallel right lines, it makes the alternate angles equal to each other.*



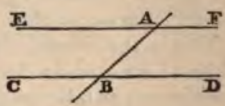
Let AB be a right line intersecting the two parallel right lines CD and EF. Then the *alternate* angles, that is, the angles EAB and ABD (or FAB and ABC) are equal to each other.

DEM. From B draw the right line BG to any point G in EF; take BH equal to GA, and on the same side of BG as GA; draw AH. Now in the triangles AGB, ABH, the sides AG, BH are equal to each other, the sides BG, HA are, by DEF. IX., equal to each other, and the side AB is common

to the two triangles. Hence, by ART. 6, the angle EAB opposite BG is equal to the angle ABD opposite AH.

Again: By ART. 9, the angles BAG, BAF are together equal to two right angles; so are the angles ABH, ABC together. Consequently BAG, BAF together are equal to ABH, ABC together*. Hence, taking away from these equals, the equal angles BAG, ABH respectively, the angle BAF remains equal to the angle ABC†. This, &c.

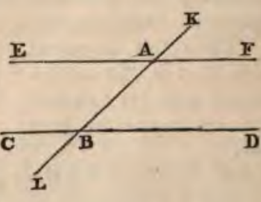
ART. 13. *If a right line intersect two parallel right lines, it makes the two internal angles on the same side of the intersecting line together equal to two right angles.*



Let AB be a right line intersecting the two parallel right lines CD and EF. Then the two *internal* angles on the same side of the intersecting line, that is, the angles FAB and ABD (or EAB and ABC) are together equal to two right angles.

DEM. By ART. 9, the angles EAB and BAF are together equal to two right angles; and by ART. preceding, ABD is equal to EAB. Hence the angles ABD and BAF together are equal to two right angles. In the same manner the angles EAB, ABC together are proved equal to two right angles. This, &c.

ART. 14. *If a right line intersect two parallel right lines, it makes each external angle equal to the farther internal angle on the same side of the intersecting line.*

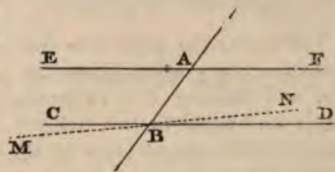


Let AB be a right line intersecting the two parallel right lines CD, EF. Then the *external* angle KAF is equal to the farther internal angle ABD on the same side of AB. Also, the other external angles KAE, LBD, LBC are respectively equal to the corresponding internal angles ABC, BAF, BAE.

* By Ax. 3. † By Ax. 4.

DEM. By ART. 11, the angle KAF is equal to the angle EAB; and by ART. 12, the angle ABD is equal to the same angle EAB. Hence the angles KAF and ABD must be equal*. In the same manner the other exterior angles are proved equal to the corresponding interior ones. This, &c.

ART. 15. *If a right line intersect two right lines, and make the alternate angles equal to each other, these two latter right lines are parallel.*



Let AB be the right line which intersects two other right lines, CD and EF, making the alternate angles EAB and ABD equal to each other. Then CD is parallel to EF.

DEM. Through the point B draw any right line, as MN, not coincident with CD. MN is not parallel to EF; for if it were *supposed* parallel to EF, then by ART. 12, the angle EAB would be equal to ABN, and therefore the angle ABD†, which is granted equal to EAB, would also be equal to ABN,

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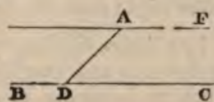
PROB. IX. *Through a given point outside a given right line to draw a right line parallel to the given one.*



Let A be the given point, BC the given line. It is required to draw through the point A a right line parallel to BC.

CONS. Draw a right line from the given point to any point D in the given line, and make the angle DAF equal to the *alternate* angle ADB, by PROB. VIII. Then the right line AF is parallel to BC.

DEM. As the right line AD meets the two right lines AF, BC, and makes the alternate angles DAF, ADB equal, the right lines AF, BC, are parallel, by ART. 15. This, &c.

This problem is also inaccurately enounced by the editors of Euclid in general; for the condition of the given point not being *in* the given line is omitted.

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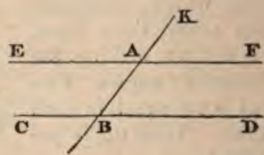
* By Ax. 3. † By Ax. 3.

which is impossible. Consequently the above supposition is false; MN is not parallel to EF. In like manner it can be proved that no other right line through the point B is parallel to EF, except CD. Hence CD is parallel to EF. This, &c. [See NOTE K.]

ART. 16. *If a right line intersect two right lines, and make the external angle equal to the farther internal angle at the same side of the intersecting line, these two latter right lines are parallel.*

Let AB be the right line intersecting two other lines EF, CD, and making the external angle KAF equal to the farther internal one ABD. Then EF is parallel to CD.



DEM. By ART. 11, the angles KAF and EAB are equal; therefore, since KAF is equal to ABD, also EAB is equal to ABD. Hence, by ART. preceding, EF is parallel to CD. This, &c.

ART. 17. *If a right line intersect two right lines, and make the two internal angles at the same side of the intersecting line together equal to two right angles, these two latter right lines are parallel.*

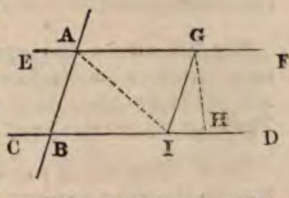
In a figure similar to the preceding, let the two internal angles FAB and ABD be together equal to two right angles. Then EF is parallel to CD.

DEM. By ART. 9, the angles FAB, EAB, together, are equal to two right angles; therefore they are equal to the angles FAB, ABD, together *. Take away from these equals the common angle FAB, and EAB remains equal to ABD †. Hence, by ART. 15, EF is parallel to CD. This, &c.

ART. 18. *If a right line intersect two parallel right lines, and another right line be drawn parallel to the inter-*

* By Ax. 3. † By Ax. 4.

sector from any point in either of the parallels, it will meet the other, if produced sufficiently; and its length between the parallels will be equal to the length of the intersector between the parallels.



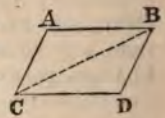
Let AB be the right line intersecting the two parallel right lines EF, CD. Then if from any point, G, in either of these parallels, EF, a right line be drawn parallel to AB, this right line will meet the other parallel, CD; and its intercepted part, between the parallels EF and CD, will be equal to the intercepted part AB.

DEM. Take BI equal to AG, and join IG. Since EF is parallel to CD, and the points A and G on the one are equally remote as the points B and I on the other, therefore, by DEF. IX. the lines AB and GI are equal. Join AI. In the triangles IBA, AGI, the side AI is common, the side BI has been taken equal to the side AG, and the side AB has been proved equal to the side GI; consequently, by ART. 6, the angle IAB, opposite to the side BI, is equal to the angle AIG, opposite to the corresponding side AG. But these are alternate angles; therefore, by ART. 15, the lines AB and GI are parallel; and therefore, by ART. 13, the two internal angles BAG, AGI are together equal to two right angles. Now the right line drawn from the point G parallel to AB must fall on the line GI. For *suppose* it to fall in any other direction, as GH; the two internal angles BAG, AGH, would, by ART. 13, be together equal to two right angles, and, therefore, equal to the angles BAG, AGI, together, which have been proved also equal to two right angles. Consequently, taking away from both the common angle BAG, there would remain the angle AGI equal to the angle AGH, the less to the greater, which is impossible. Hence, the above supposition was false; the right line drawn from the point G, parallel to AB, does not fall in any other direction than GI, and its intercepted part, GI, has been proved equal to AB. This, &c. [See NOTE L.]

LESSON IV.

ART. 19. *Right lines which join the adjacent extremities of two equal and parallel right lines are themselves equal and parallel.*

Let AB, CD be two equal and parallel right lines, whose adjacent extremities are joined by the right lines AC, BD. Then AC, BD are also equal and parallel.

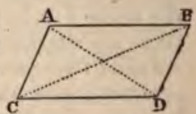


DEM. Join either pair of *opposite* extremities C and B. Now in the triangles ABC, BCD, the sides AB and CD are equal, the side BC is common, and the contained angles ABC, BCD, are equal by ART. 12. Hence, by ART. 1., AC is equal to BD; also the angles ACB, DBC, being equal by ART. 2, the lines AC, BD are paralleled by ART. 15. This, &c.

Observation. The equality of AC to BD follows immediately from DEF. IX.

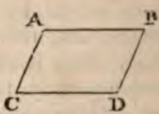
DEF. XI. A *parallelogram* is a four-sided rectilineal figure, each pair of whose opposite sides are parallel.

Thus, if AC and BD are parallel, and also AB and CD, the figure ABDC is called a parallelogram. AD and BC are called its *diagonals.*



ART. 20. *The opposite sides of a parallelogram are equal.*

Let ABDC be a parallelogram. Then AB and CD (or AC and BD) are equal.



DEM. Inasmuch as the right line AC meets the parallel right lines AB and CD, and from the point B in one of these parallels a right line BD is drawn meeting the other CD, and parallel to AC, (by DEF. XI.,) therefore BD is equal to AC, by ART. 18. In the same way it is proved that AB is equal

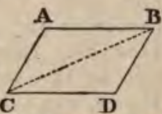
to CD; the line AB being drawn parallel to CD from a point in AC which is parallel to BD. This, &c.

ART. 21. *The opposite angles of a parallelogram are equal.*

In the preceding figure the angles ABD and ACD (or CDB and BAC) are equal.

DEM. By ART. 13, ABD and BDC together are equal to two right angles; and therefore to BDC and DCA together, which are by the same ART. also equal to two right angles. Hence, taking away BDC from both, ABD remains equal to DCA. In the same manner BAC is proved equal to CDB. This, &c.

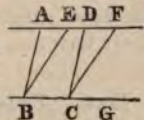
ART. 22. *A parallelogram is divided into two equal parts by its diagonal.*



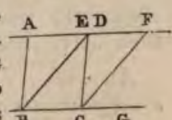
In the parallelogram ABDC, let the diagonal BC be drawn. Then the triangle ABC is equal to the triangle BDC.

DEM. By ART. 20, AB equals CD, and AC equals BD; also the side BC is common to the two triangles ABC, BDC. Hence, by ART. 7, these triangles are equal. This, &c.

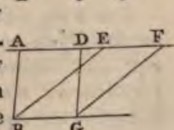
ART. 23. *Parallelograms on the same base, and between the same parallels, are equal.*



Let ABCD, EBCF, be two parallelograms on the same base BC, and between the same parallels BG, AF. Then these parallelograms are equal.



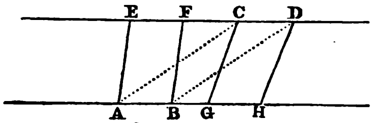
DEM. By ART. 14, the angles ABC, DCG, are equal; and also the angles EBC, FCG. Therefore, taking EBC from ABC, and FCG from DCG, the angle ABE remains equal to the angle DCF. But, by ART. 20, AB is equal to DC, and EB to FC; consequently the triangles ABE, DCF, are equal, by ART. 3. Now, if from the four-sided figure ABCF we take the triangle ABE, the parallelogram EBCF remains; and if from the same figure we take the equal triangle DCF, the parallelogram ABCD remains. Hence, as from the same quantity we successively take the equal triangles ABE, DCF,



the remainder in one case will be equal to the remainder in the other; that is, the parallelogram EBCF to the parallelogram ABCD. This, &c.

ART. 24. *Parallelograms on equal bases and between the same parallels are equal.*

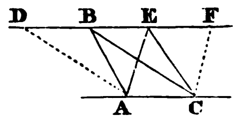
Let EABF, CGHD, be two parallelograms on equal bases AB, GH, and between the same parallels AH, ED. Then these parallelograms are equal.



DEM. Draw the right lines AC, BD. By ART. 20, CD is equal to GH, and therefore to AB, by the terms of the present ART. Consequently, by ART. 19, AC is equal and parallel to BD; and the figure ABDC is a parallelogram by DEF. XI. Now, by preceding ART. EABF is equal CABD; and by the same ART. CGHD is also equal to CABD. Hence, EABF is equal to CGHD. This, &c.

ART. 25. *Triangles upon the same base and between the same parallels are equal.*

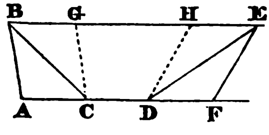
Let ABC, AEC, be two triangles on the same base AC, and between the same parallels AC, DF. Then these triangles are equal.



DEM. Draw AD parallel to CB, and CF parallel to AE. By ART. 23, the parallelograms ADBC, AEFC, are equal. Hence the triangles ABC, AEC, which by ART. 22 are their halves, are also equal*. This, &c.

ART. 26. *Triangles upon equal bases and between the same parallels are equal.*

Let ABC, DEF, be two triangles upon equal bases AC, DF, and between the same parallels AF, BE. Then these triangles are equal.

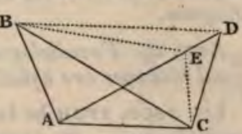


* AX. 7. *The halves of equal things are equal.*

DEM. Draw CG, DH, parallel respectively to AB, FE. By ART. 24, the parallelograms ABGC, DHEF, are equal. Hence, by ART. 22, the triangles ABC, DEF, are also equal*. This, &c.

ART. 27. *Equal triangles upon the same base, and upon the same side of it, are between the same parallels.*

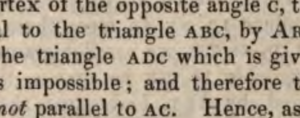
Let ABC, ADC, be two equal triangles on the same base AC, and on the same side of it. Then these triangles are between the same parallels.



DEM. Draw the right line BD from the vertex of the one triangle to the vertex of the other: BD is parallel to AC. For, *suppose* that BD is not parallel to AC, but that any other line, as BE, meeting a side AD of the triangle ADC in the point E is parallel to AC. Then, drawing from this point a right line to the vertex of the opposite angle C, the triangle AEC would be equal to the triangle ABC, by ART. 25; and consequently to the triangle ADC which is given equal to ABC. But this is impossible; and therefore the supposition is false: BE is *not* parallel to AC. Hence, as it may be proved in the same manner that no other line except BD is parallel to AC, BD is parallel to AC. This, &c. [See NOTE M.]

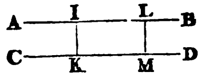
DEF. XII. Taking any side of a triangle as base, the perpendicular from the vertex to the base (produced if necessary), is called the *Altitude* of the triangle. Also, taking any side of a parallelogram as base, the perpendicular from any point in the opposite side, on the base (produced if necessary), is called the *Altitude* of the parallelogram.

Thus, if ABC, DEFG, be a triangle and parallelogram, on their respective bases AC, DG; and if BH, IK, be perpendiculars to AC, DG; then BH, IK are respectively the altitudes of ABC, DEFG.



* By Ax. 7.

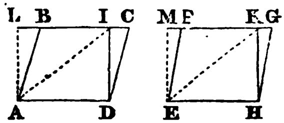
This perpendicular, it is evident, may be taken any where in the parellel EF, because it is always the same length. For if AB, CD,



be parallel, any two perpendiculars, such as IK, LM, are equal. For by ART. 17, IK and LM are parallel, and therefore, by ART. 20, they are equal, as IKML is a parallelogram.

ART. 28. *Parallelograms which have equal bases and equal altitudes are equal.*

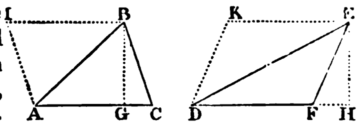
Let ABCD, EFGH be two parallelograms on equal bases, AD, EH, and with equal altitudes, DI, HK. Then these parallelograms are equal.



DEM. DRAW AL, EM parallel respectively to DI, HK; and produce IB, KF, to meet them in L and M. The figures ALID, EMKH, are parallelograms by DEF. XI. Now, if the diagonals IA, KE be drawn, the triangles IDA, KHE, are equal, by ART. I; because they have two sides, ID, DA, granted respectively equal to two, KH, HE, and the contained angles IDA, KHE also equal, by ART. 8. Consequently ALID, EMKH, the doubles of these triangles, are also equal*. Hence, inasmuch as, by ART. 23, ALID equals ABCD, and EMKH equals EFGH, the parallelograms ABCD, EFGH, are likewise equal. This, &c.

ART. 29. *Triangles which have equal bases and equal altitudes are equal.*

Let ABC, DEF, be two triangles on equal bases AC, DF, and with equal altitudes, BG, EH. Then these triangles are equal.



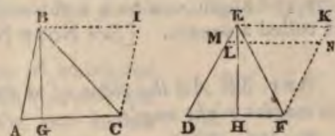
DEM. Draw AI, BI, parallel to CB, CA respectively; and

* AX. 8. *The doubles of equal things are equal.*

DK, EK parallel to FE, FD respectively. Then, by ART. preceding, the parallelograms AIBC, DKEF are equal, having equal bases AC, DF, and equal altitudes BG, EH. Hence, by ART. 22, the triangles ABC, DEF, their halves, are also equal *. This, &c.

ART 30. *Equal triangles on equal bases have equal altitudes.*

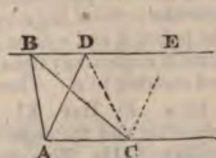
Let ABC, DEF, be two equal triangles, upon equal bases, AC and DF. Then their altitudes, BG and EH, are also equal.



DEM. For *suppose* EH to be greater than BG, and that HL is equal to BG. Draw BI, CI, parallel respectively to AC, AB; and EK, FK, parallel respectively to DF, DE; also, through the point L draw MN parallel to DF. Then, by ART. 22, the parallelograms ABIC, DEKF are equal, because their halves, the triangles, are given equal †. But, by ART. 28, the parallelogram ABIC would, *on the above supposition*, be also equal to the parallelogram DMNF; and therefore DMNF would be equal to DEKF,—a part to the whole, which is impossible. Hence this supposition is false; EH is not greater than BG. In the same way, it may be proved that BG is not greater than EH. Hence BG and EH are equal. This, &c.

ART. 31. *If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.*

Let ACED be the parallelogram on the same base AC as the triangle ABC, and between the same parallels AC, BE. Then ACED is double of ABC.



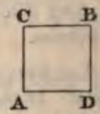
DEM. Draw the diagonal DC. By ART. 25, the triangles ABC, ADC are equal; and by

* By Ax. 7.

† By Ax. 8.

ART. 22, ACED is double of ADC. Hence, ACED is double of ABC *. This, &c.

DEF. XIII. A *square* is a parallelogram whose two adjacent sides are equal, and any of whose angles is a right angle.

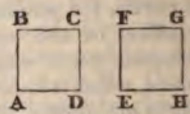


Thus, if in the parallelogram ACBD, any pair of adjacent sides AC, AD be equal, and any one angle ACB be a right angle, ACBD is called a square. [See NOTE N.]

ART. 32. *All the sides of a square are equal, and all its angles right angles.*

By DEF. XIII. and ART. 20, all its sides are equal. By ART. 13, the angle CBD is a right one, since ACB is a right one; and therefore, by ART. 21, all the angles are right. This, &c.

ART. 33. *Squares described upon equal right lines are equal.*



Let ABCD, EFGH be two squares described upon the equal right lines AD, EH. † Then these squares are equal.

This is evident. [See NOTE O.]

PROB. X. *On a given right line to describe a square.*

Let AB be the given line. It is required to describe a square on it.

CONS. At either extremity of the given line, as A, raise the perpendicular AC, by PROB. VII., and take AC equal to AB, by PROB. III. Through the points B and C draw the right lines BD, CD, parallel respectively to CA and AB, by PROB. IX. and meeting in D. Then the figure ACDB is a square.

DEM. By DEF. XI., ACDB is a parallelogram; and as AC equals AB, and the angle at A is a right angle, ACDB is a square by DEF. XIII. This, &c.

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* By Ax. 8.

† To see how a square is described upon a given right line, read Problems to number X.

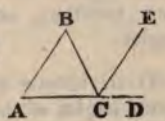
ART. 34. *If two squares be equal, their sides are equal.*

Let the squares (in fig. above) ABCD, EFGH be equal. Then also their sides AD, EH are equal.

This is evident. [See NOTE P.]

LESSON V.

ART. 35. *If any side of a triangle be produced, the external angle is equal to the two farther internal angles taken together.*



Let ABC be any triangle, of which the side AC is produced. Then the external angle BCD is equal to the two farther internal angles ABC and BAC.

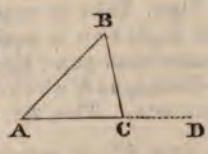
DEM. Draw CE parallel to AB. By ART. 12, the angle ABC is equal to BCE; and by ART. 14, the angle BAC is equal to ECD. Hence the two angles ABC, BAC together are equal to the two angles BCE, ECD together,—that is, to the whole angle BCD. This, &c.

ART. 36. *The external angle of any triangle is greater than either of the two farther internal angles.*

This is immediately evident from the last Article.

ART. 37. *The three internal angles of any triangle taken together, are equal to two right angles.*

In the triangle ABC, the angles ABC, BCA, CAB, taken together, are equal to two right angles.



DEM. By ART. 9, the angles BCA, BCD together are equal to two right angles; but by ART. 35, the angle BCD is equal to the two angles ABC, BAC together. Hence the angles BCA, ABC, BAC together are equal to two right angles. This, &c.

ART. 38. *Any two angles of a triangle are together less than two right angles; and if any angle of a triangle be obtuse or right, the other two are acute; also, if two angles of a triangle be equal, they are both acute.*

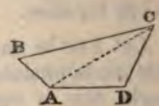
These are immediately evident from the last Article.

ART. 39. *If two triangles have two angles in the one equal respectively to two angles in the other, the third angle of the one is also equal to the third angle of the other.*

By ART. 37.

ART. 40. *The four internal angles of any four-sided rectilineal figure, taken together, are equal to four right angles.*

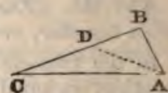
Let ABCD be a four-sided rectilineal figure. Then the angles ABC, BCD, CDA, DAB together, are equal to four right angles.



DEM. Draw the line AC. Then by ART. 37, the internal angles of the triangle ACD are together equal to two right angles; as also the internal angles of the triangle ABC. Hence, as these are the internal angles of the quadrilateral figure ABCD, they are all together equal to four right angles. This, &c.

ART. 41. *In any triangle, if one side be greater than another, the angle opposite to that greater side is greater than the angle opposite to the lesser.*

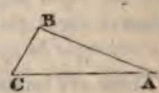
Let ABC be any triangle, having one of its sides BC greater than another BA. Then also the angle BAC is greater than the angle BCA.



DEM. Take BD equal to BA, and draw AD. By ART. 4, the angle BAD is equal to BDA; and by ART. 36, BDA is greater than BCA; therefore BAD is also greater than BCA. —Hence, BAC, which is greater than BAD, must be greater than BCA. This, &c.

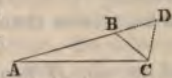
ART. 42. *In any triangle, if one angle be greater than another, the side opposite to that greater angle is greater than the side opposite to the lesser.*

Let ABC be any triangle, having one of its angles BCA greater than another BAC. Then also the side BA is greater than the side BC.



DEM. BA is not equal to BC, for in that case the angle BAC would be equal to BCA, by ART. 4; which it is *not*, by the terms of the present ART. BA is not less than BC, for, in that case, the angle BCA would be less than BAC, by preceding ART.; which it is *not*. Hence BA must be greater than BC. This, &c.

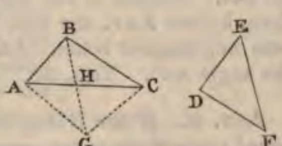
ART. 43. *Any two sides of a triangle are together greater than the third side.*



In the triangle ABC, any two of its sides, as AB, BC, are together greater than the third side AC.

DEM. Produce AB through B until the produced part BD equal BC; and draw CD. By ART. 4, the angles BDC, BCD are equal; consequently the angle ACD is greater than ADC.—Hence, by ART. preceding, the side AD is greater than the side AC; that is, inasmuch as AD is equal to AB and BC, the sides AB and BC are together greater than AC. This, &c.

ART. 44. *If two triangles have two sides of the one equal respectively to two sides of the other, but the angle contained by each pair of these sides unequal,—the base of that triangle whose given sides contain the greater angle, is greater than the base of the other triangle.*



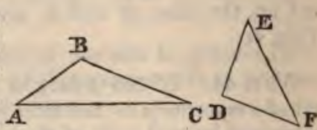
Let ABC, DEF be two triangles, having the side AB, equal to the side DE, and the side BC equal to the side EF; but the angle ABC greater than the angle DEF. Then also the base AC is greater than the base DF.

DEM. Of the two given sides AB, BC, let AB be that which is *not* the greater; and from B to AC draw BH, making with AB an angle ABH equal to the lesser angle DEF. Then, by ART. 41, since BA is not greater than BC, the angle BCA is not greater than BAC; consequently since the angle

BHC is, by ART. 36, greater than BAC, it is also greater than BCA, and therefore the side BC is greater than the side BH, by ART. 42. Produce BH below AC until BG equals BC, and draw AG, GC. Now, in the triangles ABG, DEF, since AB equals DE, and BG equals EF, and also the angle ABG equals the angle DEF,—therefore also the side AG equals the side DF, by ART. 1. But as BG is equal to BC, the angle BGC is equal to BCG; consequently the angle AGC is greater than BCG,—and therefore greater than ACG.—Hence, by ART. 42, the side AC is greater than the side AG, that is, greater than the side DF. This, &c. [See NOTE Q.]

ART. 45. *If two triangles have two sides of the one equal respectively to two sides of the other, but their bases unequal,—the vertical angle of that triangle which has the greater base is greater than the vertical angle of the other triangle* *.

Let ABC, DEF be two triangles, having the sides BA, BC respectively equal to ED, EF; but the base AC greater than the base DF. Then also the angle ABC is greater than the angle DEF.



DEM. The angle DEF is not equal to ABC, for then by ART. 1, the bases DF and AC would be equal, which they are not. Neither is the angle DEF greater than ABC; for then, by last ART., the base DF would be greater than the base AC, which it is not.—Hence the angle DEF is less than the angle ABC. This, &c.

ART. 46. *If two triangles have two angles of the one equal respectively to two angles of the other, and a side of the one triangle equal to a corresponding side of the other,—these triangles are in every respect equal to each other.*

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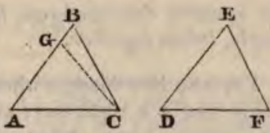
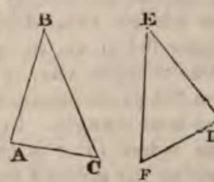
* The terms *base* and *vertical angle* are used in these two *Articles* to shorten the enunciation, and prevent confusion. See Observation on DEF. V.

CASE I. Let ABC, DEF be two triangles having the angles BAC, EDF equal; as also the angles BCA, EFD; and the side AC *between* the given angles in the one equal to the side DF *between* the given angles in the other. Then the triangles ABC, DEF are in every respect equal.



DEM. The side AB must be equal to the corresponding side DE. For *suppose* AB greater than DE, and that AG is taken equal to DE. Then, if GC were drawn, the sides AG, AC would be respectively equal to the sides DE, DF; and the angle at A is equal to the angle at D. Therefore, *on this supposition*, the angle GCA would be equal to EFD by ART. 2; and consequently to BCA,—a part to the whole, which is impossible. The above supposition is consequently false; AB is not greater than DE; and in the same manner it can be proved that DE is not greater than AB.—Hence AB and DE are equal; and therefore by ART. 3, the triangles ABC, DEF are in every respect equal, the angle at B to the angle at E, the side BC equal to the side EF, and the whole area or surface ABC equal to the whole area DEF.

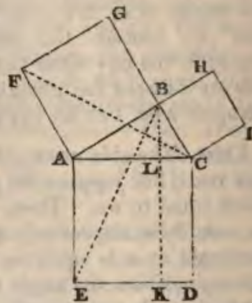
CASE II. Let ABC, DEF be two triangles having the angles BAC, EDF equal, as also the angles BCA, EFD; and the side AB (or BC) *opposite* a given angle, C, in the one triangle, equal to the side DE, (or EF,) opposite the corresponding angle F, in the other. Then the triangles ABC, DEF are in every respect equal.



DEM. By ART. 39, the angle at B is equal to that at E. —Hence, by Case I., the triangles ABC, DEF are in every respect equal,—the side AB, which is equal to DE, lying *between* the angles at A and B, which are respectively equal to those at D and E. This, &c.

ART. 47. *In any right-angled triangle the square described on the side opposite the right angle is equal to*

the squares described on the sides containing the right angle, taken together.

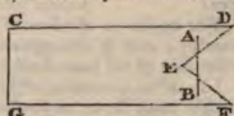


Let ABC be a triangle whose angle ABC is a right one; and let ACDE, AFGB, BHIC, be squares respectively described on the sides AC, AB, BC. Then ACDE is equal to AFGB and BHIC taken together.

DEM. Draw BE, FC; as also BK parallel to AE. In the triangles ACF, ABE, the side AF is equal to the side AB, and the side AC to the side AE, by ART. 32; also the angle FAC is equal to the angle BAE, because the angle BAC is common to both, and the angle FAB is equal to CAE, by ART. 8. Consequently, by ART. 3, the two triangles ACF and ABE are equal. But, by ART. 10, BC and BG are in the same right line, because the angles ABC, ABG are both right angles; therefore, the whole line GC being parallel to AF, the parallelogram AFGB is double of the triangle ACF, by ART. 31, and consequently double of the triangle ABE, which is equal to ACF. Likewise, as BK is parallel to AE, the parallelogram ALKE is double of the same triangle, ABE, by ART. 31.—Hence, the square AFGB and the parallelogram ALKE are equal, each being double of the same triangle. In the same way it may be proved that the square BHIC is equal to the parallelogram LCDK, by joining the points B and D, I and A. For the triangles thus formed will have two sides AC and CI respectively equal to two DC and CB; and the included angles ACI, DCB, will also be equal, as in the preceding part of the proof. Hence, by ART. 3, these two triangles will be equal; and therefore the square BHIC, which is double of one, will be equal to the parallelogram LCDK, which is double of the other. (ART. 31.)—Hence the two squares AFGB and BHIC are together equal to the two parallelograms ALKE and LCDK together; that is, to the square ACDE This, &c.

NOTES TO PART I.

NOTE A. The acknowledged superiority of Geometry as a science has tempted its professors to claim for it a perfection which can belong to nothing human. Their endeavours to exalt it into a perfectly abstract science have led to many absurdities. Thus, for example, the first principle with which the Elements of Euclid set out is either unmeaning or untrue: "A point is that which has no parts." If by the word *that* is to be understood a *magnitude,*—then it will follow from the definition in Euclid, that a point is a magnitude *which has no magnitude.* If by the word *that* is only meant (ENS) a *thing,*—then it will follow that *Spirit, Motion, Volition,* in short, whatever has no *parts,* is "a point." From the same desire to give an air of abstractness to this science, originated the definition (as it is called) of a Right or Straight line; which is said to be "that which lies evenly between its extreme points." This is a mere verbal illusion, teaching us nothing whatsoever; for we have just as clear an idea of a *straight* line as of an *even* line, of *straightness* as of *evenness,* which are indeed words of nearly the same meaning in this definition. So that the definition comes to this, viz. "A straight line is that which lies *straightly* between its extreme points,"—an evident tautology. The definition of a Plane Surface is open to still greater objections. Euclid defines it to be, "that which lies evenly between its extreme right lines;" where, together with the illusion of the word *evenly,* it may be objected that there are many plane surfaces which are not bounded by right lines, as a *circle,* &c. R. Simson gives another definition, viz. "A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies." To which it may be objected, that, in the first place, this is not a definition, but a *theorem* (if properly expressed): and in the second place, that in a plane surface like that in the annexed plate, CDEFG would not, by this definition, be a plane surface, because the right line AB does not lie *wholly* in it. The truth is, that the ideas of a Point, a Right Line, and a Plane Surface, are what logicians call *Simple Ideas,* which cannot be defined. (See *Locke's Essay,* Book III. Chap. IV.) By our sight and touch we get the ideas of straight lines and flat surfaces, nor is any definition wanting to explain them; and from these sensible ideas our mathematical ones are wholly derived. We conceive straight lines and flat surfaces as *perfectly* straight and *perfectly* flat; which, though they do not exist in nature, may exist in our minds,—and these are our mathematical ideas of the said magnitudes. They are in fact only our ideas of sensation modified and refined; so true it is that Geometry, the purest and most abstract of the sciences, is primitively derived from the commonest and humblest source of information—the senses.



The word "even" is so very general and vague, a smooth *globe* being called even, that it was thought prudent to add the words "straight" and "flat" to the description of a right line and plane surface, as perhaps

more intelligible though less refined. It is also to be observed, that the explanations of a right line and plane surface given in this Treatise are applicable to *infinite* as well as finite right lines and plane surfaces,—which the "definitions" in the Elements are not. Of a *Mathematical* Solid there is no definition given or attempted, in Euclid: his definition of a Solid "that which has length, breadth, and thickness," being evidently applicable to *any* solid. The *perfect* smoothness of its surface, or surfaces, is that which distinguishes a mathematical from a physical or common solid. Another difference, not essential however, is; that a mathematical solid is considered as the *space* which a solid would occupy, rather than as a *mass;* but the mathematical results are not at all affected by the way in which we choose to consider it.

Plane Geometry is, properly speaking, that science which treats of geometrical quantities lying in the *same plane* (as we said in our first Edition): but it is better perhaps to define it in its most usual, than its most philological acceptation.

NOTE B. In the subsequent parts of this Treatise we shall for brevity omit the words "plane" and rectilineal;" it being understood that all the lines and figures spoken of are rectilineal and plane, except the *circle,* which is plane, but not rectilineal.

The Work is divided into Lessons, not that there are *any* logical grounds for the division, but in order that each may be considered as a separate Study to be mastered completely before proceeding any farther. Every advantage, however, is taken of a change in the subject, to begin a new lesson.

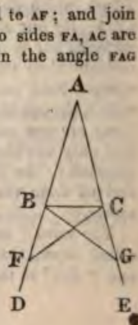
NOTE C. Articles 1, 2, and 3 form Proposition IV. in the Elements of Euclid; but besides that they are really distinct theorems, their being crowded into one enunciation confuses the mind and oppresses the memory of a reader.

NOTE D. Euclid's enunciation and proof of this theorem are as follows:

The angles at the base of an equal-sided triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be also equal.

Let ABC be a triangle, of which the side AB is equal to AC, and let the straight lines, AB, AC, be produced to D and E: the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE cut off AG equal to AF; and join FC, GB. Because AF is equal to AG, and AC to AB, the two sides FA, AC are equal to the two GA, AB respectively; and they contain the angle FAG common to the two triangles, AFC, AGB. Consequently [by ART. 1.] the base FC is equal to the base GB; as also [by ART. 2.] the angle ACF to the angle ABG, and the angle AFC to the angle AGB. But because AF is equal to AG, and AB to AC, the part BF is equal to the part CG; therefore, in the triangles, BFC, CGB, the sides BF, FC, are respectively equal to the sides CG, GB, and the angle BFC is equal to the angle CGB. Consequently [by ART. 2.] the angle FBC is equal to the angle GCB; but these are the angles below the base.

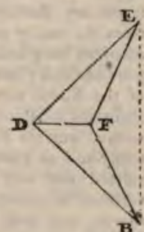


Again: In the same triangles, BFC, CGB, the angles BCF, CBG are equal [by ART. 2]. Consequently, taking these equal angles respectively from the angles ACF, ABG, which were above shown to be equal, the remaining angles ACB, ABC are equal; and these are the angles at the base. This, &c.

The above demonstration has some advantages over that given in the text; but the number and confusion of the lines and angles necessarily employed in it, have long rendered it the most difficult proposition in all Euclid to a learner. The advantage of an easy solution was to be preferred before all other less essential ones.

NOTE E. In their proofs of indirect propositions, Geometers not only omit carrying on the hypothetic phraseology, but they universally pursue a method of demonstration which is erroneous and absurd. For, taking that as true which is not true, but only *supposed* so, they tell us to do that which cannot be done, instead of telling us only to *suppose* it done. Thus, in the present theorem, R. Simson says: "let AB be the greater, and from it cut off DB equal to AC,"—referring us at the same time to PROB. III. for the method of cutting off DB. Now AB is only *supposed* greater than AC, and DB is only *supposed* cut off equal to AC; but we cannot actually cut off DB equal to AC, inasmuch as the mere supposition shows that this cannot be done. It is therefore absurd to refer us to a problem for the method of doing that which we afterwards find could not have been done. To render all such indirect proofs valid, it is only necessary, that the contrary of whatever principle we wish to assume be *supposed* true, and that we argue legitimately from such a supposition. But we are never to speak of performing actual operations on the figure, as if our supposition were really true, for they *cannot* be performed.

NOTE F. By this proof we get rid of a clumsy proposition in the Elements: which is indeed only used in order to prove this theorem, and is never afterwards employed.



The triangles ABC, DEF might be such, and applied in such a manner, that the line EB would fall outside them both, as in the annexed figure. But the demonstration given in the text would still hold good; for the angles DEF, DBF would then be the *differences* between DEB and FEB, DBE and FBE, and therefore equal, because the latter angles are respectively equal, by ART. 4. However, if we always conceive the triangles applied at their *greatest* sides, (or at any of their sides, if they are all equal), the line EB will always fall within the figure.

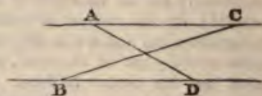
NOTE G. The definition of a Perpendicular given in Simson's Euclid begins, "*When* a straight line standing on another straight line makes the adjacent angles equal, &c.;" in which the bisection of an angle is plainly assumed, i. e. of an angle equal to two right angles.

It is of course perfectly optional to collect all the Definitions used in a work, and place them together at the beginning of it, or to introduce them according as they are wanted. But in Geometry especially there are three important advantages derivable from the latter method; so very important indeed, that it is surprising they have been overlooked by Editors in general. 1°. The memory is less burthened at the commencement

of the science; there will be less terror and disgust felt at undertaking it, when we have but a few preparatory definitions to remember, than when we have a great number to get by heart, many of which are not used for several pages. 2°. Definitions will be better understood when the learner has become gradually conversant with the ideas, phrases, and figures of Geometry. They will also be better recollected on account of the *associations* connected with them as they stand in the body of a work. 3°. The main advantage is this: that by not introducing them till they are wanted, we are enabled, by the *foregoing* articles, to prove that our definitions are possible. Thus, in our definition of a perpendicular we can prove, that one right line standing on another *may* make the adjacent angles equal to each other, and therefore that we are not assuming what might be chimerical.

NOTE H. The construction given in our first Edition for this PROBLEM might have perplexed a beginner in some cases; we therefore substitute another.

NOTE I. We must join the points A and B, C and D, which lie towards the same hand; for if we joined the points A and D, C and B, the joining lines might not be equal, though AC and BD were parallel.



The points A and C are taken *first* in the line " not the greater," because if they were taken first in the greater line, it might not be possible to take two others " equally remote" in the lesser.

NOTE J. Geometers have long confessed and regretted the imperfection of their Science in the Doctrine of Parallels. This doctrine, at least as set forth in the Elements, has been justly denominated—" the *disgrace* of Euclid." It is the only vulnerable point in the Science; and by its fatal inaccuracy, that systematic chain of rigorous demonstration, which constitutes the beauty and excellence of Geometry, is unhappily broken. Many attempts have been made to remedy this imperfection, to supply this indispensable link; but they have all been unfortunate. R. Simson, the editor of the Euclid now in use, employs two definitions, an axiom, and *five* theorems, for this purpose, yet his failure is complete. That the Author of such an humble Treatise as the present should pretend to succeed, where so many and so great men have been unsuccessful, may reasonably be considered the very height of self-delusion and self-sufficiency: but he confines his pretensions to this,—that his system of Parallels is simpler, shorter, and more strictly demonstrative, than that given in the common Euclid.

The objection to Euclid's system is this, viz.:—In order to demonstrate the properties of parallel right lines, he assumes as self-evident a principle which is *not* self-evident. This, in a Science which professes to ground itself wholly on self-evident principles, is the greatest imperfection next to a false assumption. The system brought forward in our Treatise, it is believed, assumes no principle but what is immediately self-evident to the most ordinary capacity, and is as plain to the understanding of every person as an axiom can possibly be.

It will easily be observed, by any one who compares them, how much more simply and directly the several results follow from the System now

brought forward, than from the one in common use. By the old method, also, Propositions XVI. and XVII. of Euclid [ARTICLES 36 and 38], which are properly but subordinate results of Prop. XXXII. [ARTICLES 35 and 37], were obliged to be introduced *in order to prove* Prop. XXXII.: a most unphilosophical process, inasmuch as the particular truths should always be deduced from the general one, and not *vice versâ*. By our method, Prop. XXXII., which contains the most beautiful result and the most powerful theorem of all Geometry, might be introduced so early as the 15th Article, though we postpone it to the 35th, in order to a more perfect arrangement.

NOTE K. In this demonstration it is taken for granted, that as no other right line *but* CD can be parallel to EF, therefore CD *is*. This mode of proof is common to Euclid, as in ART. 10 [Prop. XIV. Euc.] But it certainly is not a logical nor legitimate inference that CD is parallel to EF, *only* because no other line is; no more than it is a logical or legitimate inference in Art. 10, that BC is in a right line with BD, *only* because no other line is. But the validity of both these proofs (and of all others similar to them) depends on a self-evident truth in the mind of the reader, which, because it is so obvious, writers on Geometry do not think it necessary to mention. Thus, in ART. 10, it is self-evident, that there must be *some* line which, being added to BD, shall lie in the same right line with it; and as it is there proved that no other line except BC can be that line, we are warranted in concluding that BC must be it. In the present Article likewise, it is self-evident that there must be *some* line which, being drawn through the point B, shall lie parallel to EF; and as it is proved that no other line except CD can be this line, therefore CD must be it.

It may be asserted that this latter principle, taken as self-evident, is not so; and that consequently here our doctrine of parallels fails. To this it may be answered, that we conceive the principle must be self-evident to any one who has a clear notion of right lines.

NOTE L. The element in our first Edition for which the present is substituted, will be found in ART. 128, demonstrated from such principles only as are already established.

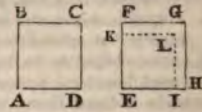
NOTE M. As BD is *supposed* not parallel to AC, if perpendiculars be dropped from B and D on AC, they would be unequal, by ART. 19. It will be most convenient therefore to draw BE from that vertex B, whose perpendicular distance from AC is supposed the least; for in this case BE will meet the perpendicular from the other vertex, D, below that vertex, and therefore also one or both of the sides AD, CD.

NOTE N. The common definition of a square is—"a four-sided figure, which has all its sides equal, and all its angles right angles." But this definition is in part superfluous; for if but *one* of the angles be a right angle, it will follow (as in the next ART.) that *all* the angles must be right angles. Therefore it is sufficient to the definition of a square, that one of its angles be granted a right angle.

NOTE O. Demonstration of ART. 33.

By ART. preceding, AB is equal to AD, and also EF to EH. Therefore, as AD and EH are equal, AB and EF must be equal. Moreover, as the angles at A and E are right angles, by preceding ART., AB and EF are the altitudes of these parallelograms (DEF. XII). Hence, by ART. 28, ABCD is equal to EFGH. This, &c.

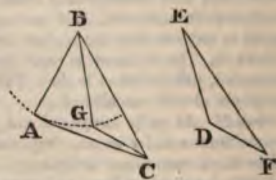
NOTE P. Demonstration of ART. 34.



Suppose EH greater than AD, and EI taken equal to AD. Also, take EK equal to EI, and draw IL parallel to EK, and KL parallel to EI. Then EKLI is a square, for it is a parallelogram whose adjacent sides are equal, and whose angle KEI is a right angle. (DEF. XIII.) Consequently, *on the above supposition*, EKLI would be equal to ABCD, by preceding ART.; and therefore to EFGH, inasmuch as ABCD and EFGH are given equal. But as a part cannot be equal to the whole, this is impossible, and the above supposition must be false. In the same way it can be proved that AD is not greater than EH. Hence AD and EH must be equal. This, &c.

NOTE Q. In order to prevent splitting this demonstration into cases, we form the equal angle ABH with that side of ABC which is "not the greater;" for if we formed it with the greater side, then the other side might fall either above, upon, or below the base AC, rendering a proof for each case necessary. Geometers add to their demonstration of this theorem a Note, proving that when the side not the greater is chosen, the point G will always fall below the base AC; they generally divide it into three cases, and use indirect proofs. But the substance of this Note is embodied in our demonstration, and a *direct* proof is given in few words.

Simson's Note in his edition of Euclid is erroneous, and totally inconclusive. He says, that, "because C is in the circumference of a circle described from the centre B with the distance BG, [*See figure*, ART. 44,] it must be in that part of it which falls above AG, the angle ABC being greater than ABG." But the very same argument would prove that G would fall below AC, *whichever* side we had formed the angle with. Thus, if BC be the greater side, and that the angle CBG equal to FED be formed with it,—as the extremity of the lesser side A is in the circumference of the circle described with the centre B and the distance BG, and as the angle CBA is greater than CBG,—it would follow from Simson's reasoning that G must fall below AC, which it evidently does not. The learned Editor was deceived by his figure.



PART II.

Of the Circle.

LESSON VI.

Observation. One extremity of a definite right line remaining fixed, if the line revolve about this point, it is evident that the other extremity will trace out a line which is every where equally distant from the fixed point; and that if the line revolve progressively to its first direction, the line which it traces out will return into itself, so as totally to include a surface.

Thus if AB be the right line, A its fixed extremity, the line BCDB will be traced out by the progressive revolution of AB about the point A, through the points C, D, to its first direction AB. Also every point of this line will be equally distant from A.

DEF. XIV. A *Circle* is a plane figure bounded by one line, such, that all right lines drawn from it to one and the same point are equal to each other.

DEF. XV. In a circle the bounding line is called the *Circumference*, and the point to which the equal lines are drawn from the circumference is called the *Centre*. [See NOTE R.]

ART. 48. *The centre of a circle falls within the circumference.*

This is evident. [See NOTE S.]

ART. 49. *A circle cannot have more than one centre.*

This is evident. [See NOTE T.]

DEF. XIV. Any right line terminated both ways in a circle is called a *Chord* of the circle, or of the arch it cuts off.

Thus in the circle (ART. 50) ABC, AB is a chord of the arch ACB, or AFB.

ART. 50. *A right line perpendicular to a chord through its middle point, will pass, if produced, through the centre of the circle.*

Let DC be perpendicular to AB at its middle point D. Then the centre of the circle ABC must be in the direction of DC.

DEM. If the chord AB pass through the centre, its middle point must be the centre (DEF. XIV.) as is plain; therefore the line DC, in *this* case, would also pass through the centre. If the chord AB do not pass through the centre, the line DC passes through it. For *suppose* the centre to lie any where out of the direction DC, as at E; and draw EA, ED, EB. Then in the triangles EDB, EDA, EB would be equal to EA, since E is the centre; DB is given equal to DA; and ED is a common side. Therefore, by ART. 6, the angle EDB would be equal to EDA, and consequently each would be a right angle by DEF. VI. But CDB is a right angle, because CD is a perpendicular. Consequently, *on the above supposition,* the angle EDB would be equal to the angle CDB, by ART. 8,—the whole to a part, which is impossible.—Hence, the above supposition is false; that is, the centre does not lie out of the perpendicular DC, but in it. This, &c.

PROB. XI. *To find the centre af a given circle.*

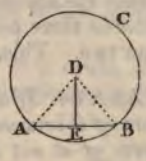
Let ABC be a circle. It is required to find the centre of it.

CONSTRUCTION. Draw any chord AC in the circle, and divide it at the point D into two equal parts, by PROB. V. Through D draw the chord BE perpendicular to AC, by PROB. VII., and divide BE into two equal parts at F. Then F is the centre of the circle.

DEMONSTRATION. By ART. 50, the chord BE passes through the centre of the circle.—Hence, by DEF. XIV., its middle point F is the centre of the circle. This was what was required by the present Article.

ART. 51. *In a circle, a right line from the centre perpendicular to a chord divides it into two equal parts.*

Let ABC be a circle, D its centre, and DE a perpendicular to the chord AB. Then AE equals EB.



DEM. Draw DA, DB. In the triangles DEA, DEB, the sides DA and DB are equal, by DEF. XIV.; and therefore also the angles at B and A, by ART. 4. Moreover the angles DEA, DEB are equal by DEF. VI.—Hence, by ART. 46, the sides EA and EB are equal. This, &c.

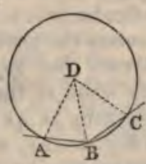
ART. 52. *In a circle, a right line through the centre dividing a chord which does not pass through the centre into two equal parts, is perpendicular to it.*

Let ABC be a circle, D its centre, and DE a right line dividing the chord AB into two equal parts at E. Then DE is perpendicular to AB.

DEM. Draw DA, DB. In the triangles DEA, DEB, the sides DA and DB are equal by DEF. XIV.; the sides EA and EB are given equal; and the side DE is common.—Hence, by ART. 6, the angles DEA and DEB are equal; that is, DE is perpendicular to AB, by DEF. VI. This, &c.

ART. 53. *A right line cannot meet the circumference of a circle in more than two points.*

Let ABC be a circle. A right line cannot meet its circumference in more than two points.



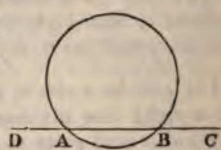
DEM. For *suppose* AC to be a right line and to meet the circumference in three points, A, B, C; and draw DA, DB, DC, from the centre D. Then as DA and DC are equal, the angles DAC, DCA, are equal, by ART. 4; but the angle DBC is greater than

DAC by ART. 36, and therefore greater than DCA. Consequently DC would be greater than DB, by ART. 42. But DC is also *equal* to DB by DEF. XIV., which is impossible. Hence the above supposition is false; AC does not meet the circle in three points; that is, it does not meet it in more than two. This, &c.

ART. 54. *If a right line meet a circle in two points, that part of it between the points lies wholly within, and those parts of it not between the points lie wholly without the circle.*

By preceding ART.; for the line cannot meet the circumference again.

DEF. XVII. A right line which meets a circle in two points but is not terminated in both, is called a *Secant*.



Thus, in the annexed figure, DB or DC is a secant. [See NOTE U.]

DEF. XVIII. A right line drawn from the centre of a circle and terminated in the circumference, is called a *Radius*.

DEF. XIX. A right line drawn through the centre of a circle, and terminated both ways in the circumference, is called a *Diameter*.

ART. 55. *In a circle, a perpendicular to a diameter, at its extremity, meets the circle in but one point.*

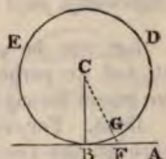
Let BCD be a circle, DB a diameter of it, and BE a perpendicular to it at its extremity B. Then BE meets the circle but at the one point B.

DEM. For *suppose* BE to meet the circle again at the point F; then if A be the centre, as AB and AF would be equal, the angles ABF, AFB, would be equal, and consequently, by ART. 38, both would be acute. But ABF is given a right angle.

Therefore the above supposition is impossible; that is, BE meets the circle but in one point B. This, &c.

DEF. XX. A right line which, however produced, meets a circle in but one point, is called a *Tangent* to that circle. [See NOTE V.]

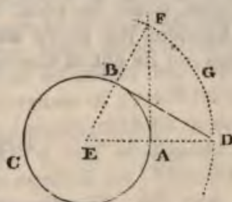
ART. 56. *If a right line be a tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent.*



Let BDE be a circle, BA a tangent to it, and CB a radius drawn to the point of contact B. Then CB is perpendicular to BA.

DEM. For *suppose* that CB is not perpendicular to BA, but that CF is the perpendicular from the centre upon it. Then, as by ART. 38, the angle CFB

PROB. XII. *From a given point without a given circle, to draw a right line which shall be a tangent to the circle.*



Let ABC be a circle, and D a point outside of it. It is required to draw from the point D a right line which shall be a tangent to the circle ABC.

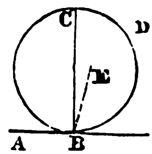
CONS. Let E the centre of the circle be found by preceding PROB.; and join DE. At the point A where the line DE meets the circle let the perpendicular AF be raised by PROB. VII. With the centre E, and the distance ED, let a circle DGF be described, meeting the perpendicular AF, produced, if necessary, in the point F. Join FE, and from the point B, where the line FE meets the circle, draw the right line BD. Then BD is a tangent from the point D to the circle ABC.

DEM. In the triangles EDB, EFA, the sides ED, EB are respectively equal to the sides EF, EA, being radii of the same circles; also the angle at E is common. By ART. 2, therefore, the angle EBD is equal to the angle EAF, that is, it is a right angle. Hence, by ART. 55, the right line DB is a tangent. This, &c..

Obs. It is evident that from the same point outside a circle two tangents may be drawn to the circle.

would be greater than CBF, the side CB would be greater than CF, by ART. 42. Consequently the radius CG, which is equal to CB, would be greater than CF; a part greater than the whole which is impossible. Therefore, the above supposition is false; CF is not perpendicular to BA, nor is any other line but CB. This, &c.

ART. 57. *If a right line be a tangent to a circle, the perpendicular to it at the point of contact will, if produced sufficiently, pass through the centre.*

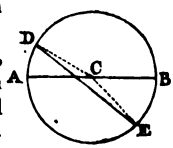


Let BDC be a circle, BA a tangent to it, and BC perpendicular to BA. Then BC passes through the centre of BDC.

DEM. For *suppose* BC not to pass through the centre, but that the centre lies outside of this line, as at E. Then, by ART. preceding, the radius EB would be perpendicular to BA, and the angle EBA a right angle. Consequently, as BC is granted perpendicular to BA, the angles EBA, CBA, being both right angles, would be equal, by ART. 8,—which is impossible. Therefore the above supposition is false; the centre E does not lie out of the right line BC. This, &c.

LESSON VII.

ART. 58. *The diameter of a circle is the greatest chord which can be drawn in it.*



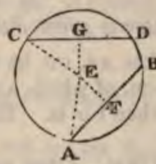
Let ADB be a circle, AB a diameter of it, and DE any other chord in it. Then AB is greater than DE.

DEM. DRAW CD, CE. By ART. 43, CD and CE together are greater than DE; but CD and CE together are equal to CA and CB together (DEF. XIV.)— Hence AB is greater than DE. This, &c.

DEF. XXI. Right lines are said to be *equally distant* from the centre of a circle when the perpendiculars drawn to them from the centre are equal. And one right line is said to be farther from the centre than another when the perpendicular on the former is greater than that on the latter.

ART. 59. *Chords equally distant from the centre of a circle are equal.*

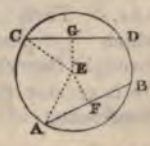
Let AB, CD, be two chords of the circle ABC equally distant from the centre E. Then AB equals CD.



DEM. Draw the radii EA, EC; also the respective perpendiculars EF, EG, which by DEF. XXI. are granted equal. By ART. 47, the square of EA is equal to the square of EF and the square of AF together: likewise the square of EC is equal to the square of EG and the square of CG together. Consequently, as EA equals EC,—and, therefore, by ART. 33, the square of EA equals the square of EC,—the square of EF and the square of FA together are equal to the square of EG and the square of CG together. Therefore, if from these equal quantities we take respectively the squares of EF and of EG, (which squares are equal by ART. 33), the square of AF will remain equal to the square of CG; and consequently AF is equal to CG, by ART. 34.—Hence, as AF is half of AB, and CG is half of CD, by ART. 51,—AB is equal to CD. This, &c.

Art. 60. *Equal chords in a circle are equally distant from the centre.*

In the circle ABDC, let the chords AB, CD, be equal. Then they are equally distant from the centre.



DEM. Draw EA, EC; also the perpendiculars EF, EG, as before. Then AB, CD, being given equal, their halves (ART. 51), AF, CG are also equal,—and therefore the squares of AF and of CG are equal, by ART. 33. But the square of AF and the square of EF together

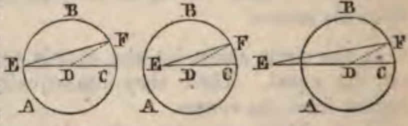
are equal to the square of EA; also the square of CG and the square of EG together are equal to the square of EC, by ART. 47. Therefore the square of AF and square of EF together are equal to the square of CG and the square of EG together. From these equals, take away respectively the equal squares of AF and of CG, and the square of EF will remain equal to the square of EG. Hence, by ART. 34, EF is equal to EG; that is, the chords AB and CD are equally distant from the centre. This, &c.

ART. 61. *In a circle the chord which is nearer to the centre is greater than that which is farther off.*

Let AB, CD, be two chords of the circle ABCD, of which AB is nearer to the centre E, than CD is. Then AB is greater than CD.

DEM. Draw EF, EG respectively, perpendicular to AB, CD; and since, by DEF. XXI. EG is greater than EF, take EH equal to EF, and draw through H the chord IK perpendicular to EA. By ART. 59, IK is equal to AB. But IK is greater than CD by ART. 44; for in the triangles IEK, CED, the two sides IE, EK are respectively equal to the two CE, ED, and the angle IEK is greater than CED.—Hence AB also is greater than CD. This, &c.

ART. 62. *From any point which is not the centre of a circle, the greatest right line that can be drawn to the circumference is that which actually passes through the centre.*

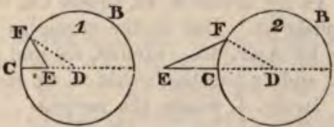


Let E be any point different from D, the centre of the circle ABF. Then EC which passes through the centre is greater than any other line, as EF, drawn to the circumference.

DEM. Draw DF. By ART. 43, ED and DF together are greater than EF; but ED and DF together are equal to ED

and DC together (DEF. XIV.) Hence EC is greater than EF. This, &c.

ART. 63. *From any point which is not the centre of a circle, the least right line that can be drawn to the circumference is that which does not, but which would, if produced, pass through the centre.*

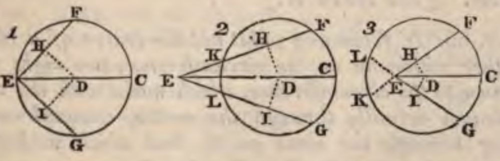


Let E be any point different from D, the centre of the circle CFB. Then EC, which would, if produced, pass through the centre D, is less than any other line, such as EF, drawn to the circumference.

DEM. Draw DF. By ART. 43, EF and ED (in fig. 1) together are greater than DF, and therefore greater than DC. Hence, taking away the line DE, which is common, EF remains greater than EC *. In fig. 2, EF and FD together are greater than ED. Hence, taking away FD and CD (which are equal) from both respectively, EF remains greater than EC *. This, &c.

ART. 64. *If from any point not the centre of a circle two right lines be drawn to the circumference, which make with that drawn through the centre, equal angles opening towards the same parts, these two lines are equal.*

Let E be any point different from D, the centre of the circle GFC; and EF, EG, two right lines making with EC

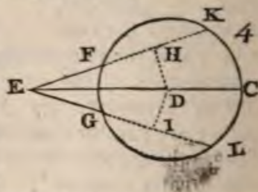


equal angles, FEC, GEC, opening towards the same hand. Then EF and EG are equal.

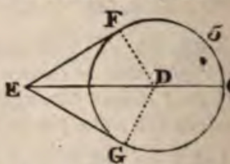
* Ax. 9. *If from unequal things we take equal things, the remainder are unequal.*

DEM. Draw the respective perpendiculars DI, DH. In the triangles DHE, DIE, since there are two angles DHE, DEH, respectively equal to two, DIE, DEI, and the side DE common,—the side DH also equals the side DI, by ART. 46. Consequently, the chords on which these perpendiculars fall are equal by DEF. XXI. That is, in fig. 1, EF is equal to EG.

In fig. 2 and 3, FK equals GL, therefore by ART. 51, HF equals GI. But in the triangles DHE, DIE, HE is equal to IE by ART. 46. Hence, HF and HE together are equal to GI and IE together; that is, FE is equal to GE.



In fig. 4, where the perpendiculars DH, DI are let fall on the produced parts of EF and EG, these parts FK, GL, are equal as before, and therefore their halves also, HF, IG, by ART. 51. But in the triangles DHE, DIE, HE is equal to IE, by ART. 46.—Hence, taking away from these equals the equals HF and IG, respectively, the remainders, *i. e.* EF and EG, are equal.

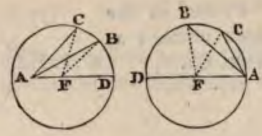


If EF and EG should be both tangents, as in fig. 5, then drawing DF, DG, in the triangles DEF, DEG, the angles DGE, DFE, are equal by ART. 56. Also, the angles DEG, DEF are granted equal, and the side DF is equal to the side DG.—Hence, by ART. 46, EF is equal to EG. This, &c. [See NOTE W.]

ART. 65. *If from any point not the centre of a circle, but either within or on the circumference, two right lines be drawn to the circumference, which make with the right line drawn actually through the centre, unequal angles opening towards the same parts, that which makes the smaller angle is greater than the other* *.

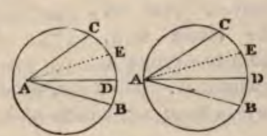
* It is evident that the line making the smaller angle will be *nearer* to the line passing through the centre than the other; but it is more analogous with ART. 64 to enounce the proposition in this way, than by using the word "nearer," as is generally done.

Let A be a point different from F, the centre of the circle DBC, and either within the circumference or on it. Then AB, which makes with AD the angle BAD, is greater than AC, which makes with AD a greater angle.



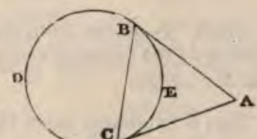
DEM. If AB and AC be on the same side of AD, as in the above figure,—drawing the radii FB, FC, in the two triangles AFB, AFC, the sides FB and FC are equal, the side FA is common, but the angle AFB is greater than AFC. Hence, by ART. 44, the side AB is greater than AC.

Again: If AB and AC be at different sides of AD, as in this figure, draw AE making the angle DAE equal to DAB, but at different sides of AD. Then, by ART. 64, AE and AB are equal. But, by preceding part of this demonstration, AE is greater than AC. Hence also AB is greater than AC. This, &c.



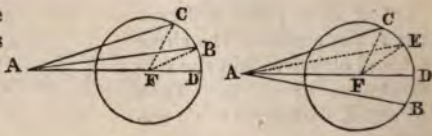
DEF. XXII. If from a point outside a circle, two tangents be drawn to the circle, and the points of contact be joined by a right line, that part of the circumference lying within the triangle thus formed is called the *Convex*, and that part of the circumference lying without this triangle is called the *Concave* part of the circumference, with respect to the given point.

Thus, if AB and AC be tangents to the circle BCD, the part BEC is the convex, and BDC the concave part of the circumference, with respect to the point A.



ART. 66. *If from a point outside a circle two right lines be drawn to the concave part of the circumference, which make with the right line through the centre unequal angles, that which makes the smaller angle is greater than the other.*

Proved in the same manner as Art. 65.



Art. 67. *If from a point outside a circle two right lines be drawn to the convex part of the circumference, which make with the right line through the centre unequal angles, that which makes the smaller angle is less than the other.*

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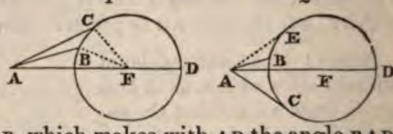


Let AB and AC be drawn to the convex part of the circle BCD, from the point A without it. Then AB, which makes with AD the angle BAD, is less than AC, which makes with AD a greater angle.

Dem. If AB and AC be on the same side of AD, as in fig. 1, drawing the radii FB, FC, in the two triangles AFB, AFC, the sides FB and FC are equal, the side FA is common, but the angle AFB is less than the angle AFC. Hence, by Art. 44, the side AB is less than the side AC.

Again: if AB and AC be on different sides of AD, as in fig. 2, draw AE making the angle DAE equal to DAC, but at different sides of AD. Then, by Art. 64, AE and AC are equal. But, by preceding part of this demonstration, AB is less than AE;—hence also AB is less than AC. This, &c.

Art. 68. *More than two equal right lines cannot be drawn to the circumference of a circle from any one point but the centre.*

For, if from any point but the centre three right lines be drawn to the circumference, either one will pass in the direction of the centre, and therefore be greater or less than each of the others, by Articles 62 and 63; or none of them will lie in the direction of the centre, and therefore two of them will lie at the same side of that direction, which two will be unequal, by Articles 65, 66, and 67.

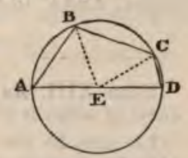
LESSON VIII.

DEF. XXIII. A rectilineal figure is said to be *inscribed* in a circle when the vertices of all its angles are situated in the circumference of that circle.

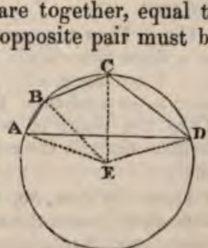
ART. 69. *The opposite angles of a four-sided rectilineal figure inscribed in a circle, are together equal to two right angles.*

Let ABCD be a four-sided rectilineal figure inscribed in the circle ACD. Then the angles ABC, ADC, together, as also the angles BAD, BCD, together, are equal to two right angles.

DEM. *Case* 1. When the centre is at E *within* the figure, draw the radii EA, EB, EC, ED. Then, because these radii are equal.—by ART. 4 the angles EBA, EBC, EDA, EDC equal, respectively, the angles EAB, ECB, EAD, ECD; that is, (as the first four angles make up ABC and ADC, and the last four angles make up DAB, DCB,) the opposite angles, ABC, ADC, together, are equal to the opposite angles, DAB, DCB, together. Hence, as these two pair of angles are together equal to four right angles by ART. 40, each opposite pair must be equal to two right angles.



Case 2. When the centre E is *on one side* of the figure, draw the radii EB, EC. Then, by ART. 4, the angles EBA, EBC, EDC, are respectively equal to the angles EAB, ECB, ECD; that is, the opposite angles, ABC, EDC, together are equal to the opposite angles EAB, BCD, together. Hence, as these two pair of angles are together, equal to four right angles, by ART. 40, each opposite pair must be equal to two right angles.

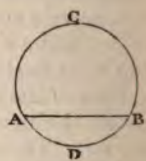


Case 3. When the centre E is *without* the figure, draw the radii EA, EB, EC, ED. Then, by ART. 4, the angles EBA, EBC, EDC, are respectively equal to the angles EAB, ECB, ECD; that is, ABC, EDC together are equal to EAB, BCD together. But

EDA and EAD are also equal, by ART. 4; therefore, taking away these equal quantities from either side, there will remain the opposite angles ABC and ADC together, equal to the opposite angles DAB and BCD together. Hence, as these two pair of angles are together equal to four right angles, by ART. 40, each opposite pair must be equal to two right angles. This, &c. [See NOTE X.]

DEF. XXIV. If any geometrical figure or magnitude be divided into two or more parts, these parts are called *Segments.*

Thus if the circle BCD be divided by the right line AB, the parts ACBA, ADBA, are called segments *of the circle;* and ACB, ADB, segments *of the circumference.*

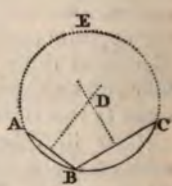


Obs. Segments of the circumference are usually called *arches* or *arcs.*

Segments of a circle may be made by *any* kind of line dividing the circle into two parts; but in Plane Geometry the only segments of a circle considered are those made by a *right* line so dividing the circle. From this circumstance the name "segment of a circle" has been appropriated solely to figures included by arches and their chords, as ACBA, ADBA, in the diagram above.

PROB. XIII. *An arch of a circle being given, to describe the circle of which it is the arch.*

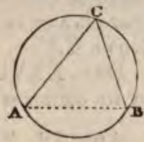
Let ABC be the given arch. It is required to describe the circle of which it is the arch.



CONS. From any point in the given arch, as B, draw two chords of the arch BA and BC. From the middle point of each chord raise a perpendicular to it by PROB. VII. Then the point D, where these perpendiculars meet, is the centre of the required circle.

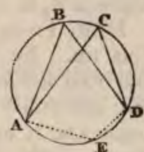
DEM. By ART. 50, the perpendicular through the middle of each chord passes through the centre. Hence, the point where these perpendiculars intersect must be the centre. This, &c.

DEF. XXV. When an angle has its vertex in an arch of a circle, and its sides terminated in the extremities of that arch, this angle is called the *Angle in a segment* of the circle.



Thus, the angle at C is the angle in the segment of the circle ACBA.

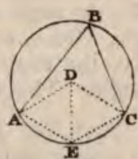
ART. 70. *The angles in the same segment of a circle are equal.*



Let ABD, ACD, be two angles in the same segment ABCD. Then they are equal.

DEM. Draw from any point E in the other segment of the circumference, the right lines EA, ED. By preceding ART. the angles at E and B are together equal to two right angles; as also the angles at E and C taken together. Therefore the angles at E and B together are equal to those at E and C together. Hence, taking from both sides the common angle at E, the angle at B remains equal to the angle at C. This, &c.

ART. 71. *The angle in the segment of a circle is half of the external angle at the centre whose sides are terminated in the extremities of the corresponding arch.*



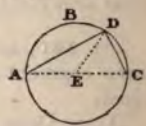
Let ABC be an angle in the segment ABCA, and ADC the external angle at the centre. Then ABC is half of ADC.

DEM. Draw the radius DE dividing the angle ADC into two equal parts, and join AE, CE. Since in the triangles ADE, EDC, the two sides AD, DE, are equal to the two ED, DC, and also the angles contained by these sides are equal, the angle DEA must be equal to the angle DCE, by ART. 2. But in the triangle EDC, the three angles DCE, DEC, EDC, are together equal to two right angles, by ART. 37; therefore also the angles DEA, DEC, EDC, are together equal to two right angles. Consequently the angles DEA, DEC, EDC, or AEC, EDC, together, are equal

to the angles AEC, ABC, together; these being, by ART. 69, *also* equal to two right angles. Hence, taking away the common quantity AEC from both, EDC remains equal to ABC; that is, ABC is equal to *half* the angle ADC. This, &c. [See NOTE Y.]

DEF. XXVI. The segments into which a circle is divided by its diameter are called *Semicircles.*

ART. 72. *The angle in a semicircle is a right angle.*

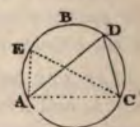


Let ABC be a semicircle. Then any angle in it, as ADC, is a right angle.

DEM. From the centre E draw the radius ED. By ART. 4, the angles EDA and EAD are equal; as also the angles EDC and ECD. Consequently, the whole angle ADC is equal to the two others DAC, DCA, together. Hence, as the three angles together are equal to two right angles, (ART. 37,) ADC must be equal to *one* right angle. This, &c. [See NOTE Z.]

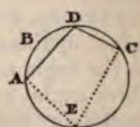
ART. 73. *The angle in a segment greater than a semicircle is less than a right angle.*

Let ABC be a segment greater than a semicircle. Then ADC is less than a right angle.



DEM. Draw the diameter CE; join EA and CA. By preceding ART., the angle EAC is a right angle: therefore, by ART. 38, the angle AEC is less than a right angle. Hence, as ADC equals AEC, by ART. 70, the angle ADC is less than a right angle. This, &c.

ART. 74. *The angle in a segment less than a semicircle is greater than a right angle.*



Let ABC be a segment less than a semicircle. Then ADC is greater than a right angle.

DEM. Join A and C with any point E in the other segment. By ART. 69, the angle D with the angle E is equal to two right angles. Hence, as the angle E by

last ART. is less than a right angle, the angle D must be greater than a right angle. This, &c.

ART. 75. *If a tangent and a chord of a circle be drawn from the same point, the angle between them is equal to the angle in the alternate segment.*

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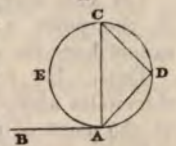

Let AB and AC be a tangent and chord through the same point A of the circle AED. Then the angle CAB is equal to the angle CDA in the alternate segment CDA.

DEM. If the chord AC pass through the centre, as in fig 1, the angle CDA is a right angle, by ART. 72; and by ART. 56, the angle CAB is also a right angle. Hence, by ART. 8, these angles are equal.

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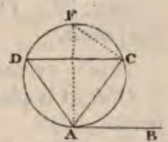

Again, if the chord AC do not pass through the centre, but falls as in fig. 2, draw AF through the centre, and join FC. By ART. 72, the angle ACF is a right one, and therefore by ART. 37, the two angles AFC, FAC, are altogether equal to a right angle. But by ART. 56, the angle FAB is a right angle; and therefore equal to the angles AFC, FAC, together. Consequently, taking away from both sides the common angle FAC, there remains the angle CAB equal to AFC. Hence, as AFC, CDA are equal, by ART. 70, the angle CAB is equal to the angle CDA.

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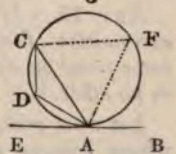


Finally: if the chord AC make an obtuse angle CAB with AB, as in fig. 3, then the angle CAE will be acute. Draw AF, CF, to any point F in the segment alternate to CAE, and by the preceding part of the demonstration, the angle CAE will be equal to the angle CFA. But by ART. 69, the angles CFA and CDA together are equal to two right angles; therefore also the angles CAE and CDA are together equal to two right angles, that is, to the angles CAE and CAB together, by ART. 9. Taking away from both the common angle CAE, there remains the angle CDA equal to the angle CAB. This, &c.

LESSON IX.

Of Circles.

ART. 76. *If two different circles meet one another, they cannot have the same centre.*

Let ABC, ADE, be two circles meeting in the point A. Then these circles cannot have the same centre.

DEM. Let F be the centre of one of the circles ADE; draw the right line FA to the point of meeting, A, and another right line FDB intersecting both circles at different points, D and B. Then as FD is equal to FA, and FB *unequal* to FD, FB is also unequal to FA. Hence F cannot be the centre of the circle ABC. Which, &c.

ART. 77. *One circle cannot meet another in more than two points.*

This is evident. [See NOTE AA.]

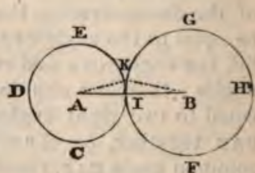
Obs. It is plain that one circle *may* meet another in two points.

ART. 78. *If one circle meet another in two points, one portion of the former will be wholly within and the other wholly without the latter circle.*

This is evident. [See NOTE BB.]

DEF. XXVII. When one circle meets another in two points, it is said to *cut* it.

ART. 79. *If two circles having their centres at the two extremities of a given finite right line pass through the same point on that finite line, they meet in that point, but in no other.*



Let CDE, FGH, be two circles whose centres are at the opposite extremities of the right line AB, and whose circumferences pass through the same point of the line I. Then these circles meet at no other point but I.

DEM. It is plain that both passing through I, they meet at that point. But they meet at no other. For on the *supposition* that they meet at a second point K, the right lines AK, BK, drawn from the centres A and B to that point, would be together greater than AB by ART. 43. But as AK equals AI, and BK equals BI, (being radii of the same circles,) AK and BK together are *not* greater than AB, but equal to it. Hence, the above supposition is false; that is, the circles do not meet in a second point. Which, &c.

DEF. XXVIII. Circles which meet in but one point are said to touch each other. [See NOTE CC.]

Obs. Hence it is evident that if two circles touch, one must be either wholly within or wholly without the other, —else they would meet in a second point.

DEF. XXIX. The point where two circles touch is called the *point of contact.*

ART. 80. *If two circles touch, the right line joining their centres, if produced, will pass through the point of contact.*

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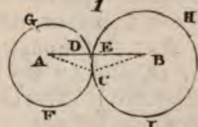

Let CFG, CHI, be two circles touching at C. Then the right line AB joining their centres will pass through C.

DEM. For if we *suppose* AB not to pass through C, let the radii CA and CB be drawn. Then, by ART. 43, in fig. 1, AC and CB together are greater than AB; that is, AD and BE (which are respectively equal to AC and BC, as radii of the same circles) would be together greater than AB,—a part greater than the whole, which is impossible. In fig. 2, BA and AC are, by ART 43, together greater than BC; that is, than BE (which is equal to BC, as radii of the same circle CHI). Consequently, taking away the common quantity BA, there would remain AC greater than AE. But AC is equal to AD (being radii of the same circle CFG); and therefore AD would be greater than AE,—a part than the whole, which is impossible. Hence the above supposition is false; that is, AB passes through C. Which, &c.

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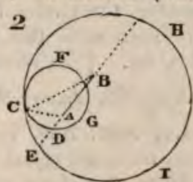


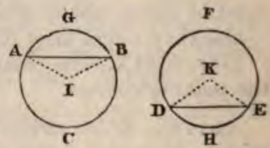
DEF. XXX. Equal circles are those which, if applied centre to centre, would exactly coincide with each other.

ART. 81. *Equal circles have equal diameters.*

For if the centres were applied, the circumferences would coincide, by preceding DEF.; therefore any diameter of the one circle would coincide exactly with a diameter of the other, and consequently be equal to it. This, &c. [See NOTE DD.]

ART. 82. *In equal circles equal chords cut off equal arches.*

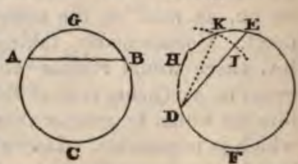
Let ABC, DEF, be two equal circles, whose chords AB and DE are equal. Then also the arches AGB and DHE are equal.



DEM. If the chords be diameters, the arches will be halves of equal circles, and therefore equal. If the chords be not diameters, draw the radii IA, IB, and KD, KE; and by ART. 6, the angle at K is equal to the angle at I, because IA, IB, and AB are respectively equal to KD, KE, and DE. (ART. 81.) Therefore if the centres were so applied that KD would fall upon IA, and KE upon IB, the chord DE would exactly coincide with the chord AB, because the radii of both circles are equal. The circumferences would also coincide, the circles being equal, by DEF. XXX.; and the part DHE with the part AGB, which are therefore equal. This, &c. [See NOTE EE.]

ART. 83. *In equal circles equal arches have equal bases.*

Let ABC, DEF, be two equal circles, whose segments AGB, DHE, are equal. Then also AB and DE are equal.



DEM. For if AB and DE be *supposed* unequal, and that one of them, as DE, were the greater: and that DI a portion of DE were equal to AB; and that with D as centre and DI as distance

a circle cutting DHE in K were described, and that finally DK were drawn. Then as DI, or DK, and AB are supposed equal, the arch DHK is, by ART. preceding, equal to the arch AGB, and therefore to the arch DHE,—a part to the whole, which is impossible. Hence the above supposition is false; that is, DE and AB are not unequal. This, &c.

Obs. It is plain that these two latter Articles are true, as well of the segments of the *circles*, as of the segments of the *circumferences*.

ART. 84. *In equal circles, equal angles, whether they be at the centres or the cirucmferences, stand upon equal arches.*

Let ABI, CDK, be two equal circles, having the angles G and H at the centres as in fig. 1, —or the angles E and F at the circumferences as in figs. 2, 3, 4,—equal. Then the arches AIB and CKD are equal.

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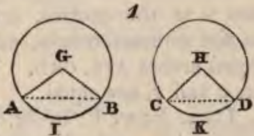

DEM. PART I. In fig. 1, draw AB and CD. Then in the triangles AGB, CHD, the sides AG, GB, are respectively equal to the sides CH, HD, (ART. 81,) and the angles at G and H are given equal. Hence, by ART. 1, AB is equal to CD; that is, by ART. 82, the arches AIB and CKD are equal.

PART II. In fig. 2, the equal angles E and F at the circumference being acute, the angles G and H at the respective centres, being their doubles, by ART. 71,—are also equal. Hence, by PART I., the arches AIB and CKD are equal.

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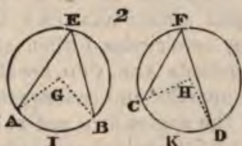

PART III. In figs. 3 and 4, the angles E and F at the circumference being either right or obtuse ones, let both be divided into two equal parts, AEI, BEI, and CFK, DFK, which will then be acute.

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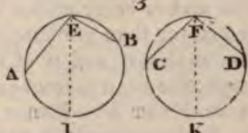

Also the angle AEI will be equal to the angle CFK, and the angle BEI to the angle DFK, as the whole angles themselves are given equal. Hence by PART II. the arches on which the angles AEI, CFK stand, will be equal; and also the arches on which the angles BEI, DFK stand; therefore the whole arch AIB will be equal to the whole arch CKD. These were the assertions, &c.

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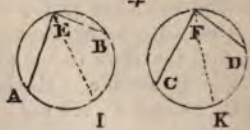

ART. 85. *In equal circles, the angles which stand upon equal arches are equal, whether they be at the centres or the circumferences.*

Let AEB, CFD, be two equal circles, with the angles G and H at the centres, as in fig. 1,—or the angles E and F at the circumferences, as in figs. 2, 3, 4,—standing upon equal arches AIB, CKD. Then also these angles at G and H, E and F, are equal.

DEM. PART I. In fig. 1, as the sides AB and CD are equal, by ART. 83; and as AG, GB are respectively equal to CH, HD, by ART. 81,—the angles at G and H are equal, by ART. 6.

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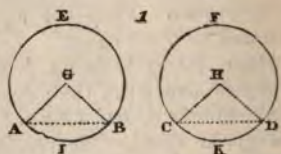

PART II. In fig. 2, (the equal arches AIB, CKD being semicircles,) then the segments AEB, CFD, are also semicircles and consequently the angles at E aud F, by ART. 72, are both right ones. Hence, by ART. 8, they are equal.

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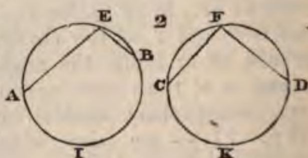

PART III. In figs. 3 and 4, (the arches being either less or greater than semicircles,) the angles G and H at the respective centres are equal, as in PART I. Hence in fig. 3, the angles at E and F

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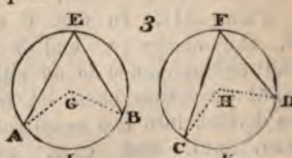
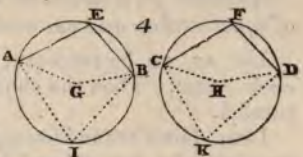


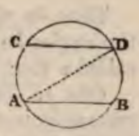


are equal, because they are halves of those at G and H, by ART. 71. Also for the same reason, the angles at I and K, in fig. 4, are equal. But the angles at I and E are together equal to two right angles, by ART. 69, and therefore equal to the angles at K and F together. Hence, taking away from both sides the equal angles at I and K, the angles at E and F remain equal.

These were the assertions, &c. [See NOTE FF.]

ART. 86. These latter four articles, it is evident, are true for the *same* circle as well as equal ones.

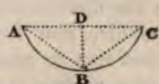
ART. 87. *In a circle parallel chords intercept equal arches.*



Let AB, CD, be two parallel chords in the circle ABDC. Then the arches AC, BD, are equal.

DEM. Draw AD, and the angles CDA, DAB are equal, by ART. 12. Hence, by ART. 86 and 84, the arches AC and BD are equal. This, &c.

PROB. XIV. *To divide a given arch of a circle into two equal parts.*



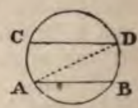
Let ABC be a given arch. It is required to divide it into two equal parts.

CONS. Draw the right line AC, joining the extremities of the given arch. Divide AC equally at the point D, by PROB. V., and from the point D raise DB perpendicular to AC, by PROB. VII. Then B will be the middle point of the arch ABC.

DEM. Draw the right lines AB and CB. In the triangles ADB, CDB, since the side DB is common, and since AD is equal to DC, and the angle ADB equal to the angle CDB, by construction,—therefore by ART. 1, AB is equal to BC. Hence, by ART. 86 and 82, the arch AB is equal to the arch BC. This, &c.

ART. 88. *In a circle, the chords joining the extremities of equal arches, and not intersecting, are parallel.*

Let AC, BD, be two equal arches of the circle ABDC. Then the chords AB, CD, are parallel.



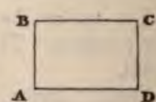
DEM. Draw AD, and the angles CDA, DAB, are equal by Art. 86 and 85. Hence by ART. 15, the chords AB and CD are parallel. This, &c. [See NOTE GG.]

LESSON X.

Of Rectangles.

DEF. XXXI. A *Rectangle* is a right angled parallelogram.

Thus, in the parallelogram ABCD, if the angles be right ones, then ABCD is called a Rectangle.

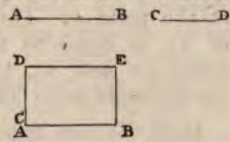


Obs. 1. When we wish to specify a rectangle, instead of specifying the four letters at its angles, we often specify merely the *two* letters at its opposite angles. Thus, in the above figure, instead of saying the rectangle ABCD, we say the rectangle AC, or the rectangle BD,—for brevity's sake. Parallelograms of any kind are often specified in the same way.

Obs. 2. The surface of a rectangle, as AC, is plainly limited in the direction from AB to CD by the length of AD; and in the direction from AD to BC by the length of AB. This gives rise to another very usual way of specifying a rectangle, namely, by help of any two of its adjacent sides. Thus we specify the rectangle AC, by saying—the rectangle *under* the sides AB and AD; or the rectangle under AD and DC. It is used merely to save the trouble of specifying the four sides.

Also: when we make use of the expression—" the rectangle under two lines,"—these lines being perhaps separate, we mean the rectangle which *might be formed* with one of

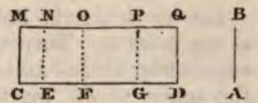
these lines as base, and a line equal to the other as the side adjacent. Thus, if AB, CD, be two right lines, when we say—"the rectangle under AB and CD," we mean the rectangle ADEB, whose adjacent sides are AB and CD,—or (which is the same thing) lines equal to them.



It is strongly recommended to the reader to fix in his mind and memory the substance of these observations before he proceeds a step farther. In them alone is centered the whole difficulty of the Doctrine of Rectangles.

ART. 89. *If there be two right lines, one of which is divided into any number of parts, the rectangle under the two lines is equal to the sum of the rectangles under the undivided line and the several parts of the divided line.*

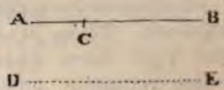
Let AB, CD, be any two right lines of which CD is divided into any number of parts, CE, EF, FG, GD. Then the rectangle under AB and CD is equal to the sum of the rectangles under AB and CE, AB and EF, AB and FG, AB and GD.



DEM. Let CQ be a rectangle whose side CM adjacent to CD is equal to AB. Then the rectangle CQ is, by preceding Observation, the rectangle under CM, or AB, and CD. Draw EN, FO, GP, parallel to CM; then CN, EO, FP, GQ, are all rectangles, whose sides EN, FO, GP, are each equal to CM, by ART. 20, and therefore to AB. But it is evident that the rectangle CQ is equal to the sum of the rectangles CN, EO, FP, GQ; that is, the rectangle under AB and CD is equal to the sum of the rectangles under AB and CE, AB and EF, AB and FG, AB and GD. This, &c.

ART. 90. *If a right line be divided into any two parts, the square of the whole line is equal to the sum of the rectangles under the whole line and each of the parts.*

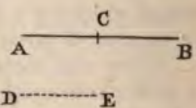
Let AB be a right line divided at the point C. Then the square of AB is equal to the rectangle under AB and AC, together with the rectangle under AB and CB.



DEM. Take a right line DE equal to AB, and by ART. preceding, the rectangle under DE and AB is equal to the sum of the rectangles under DE and AC, DE and CB. But the rectangle under DE and AB is equal to the square of AB (DEF. XIII.); hence the square of AB is equal to the sum of the rectangles under DE and AC, DE and CB, that is, to the sum of the rectangles under AB and AC, AB and CB. This, &c.

ART. 91. *If a right line be divided into any two parts, the rectangle under the whole line and either part is equal to the square of this part together with the rectangle under the parts themselves.*

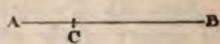
Let AB be a right line divided at any point C. Then the rectangle under AB and AC is equal to the square of AC, together with the rectangle under AC and CB.



DEM. Take the right line DE equal to AC, and by ART. 89, the rectangle under DE and AB is equal to the sum of the rectangles under DE and AC, DE and CB. But the rectangle under DE and AC is equal to the square of AC (DEF. XIII.); hence the rectangle under DE and AB, that is, the rectangle under AC and AB, is equal to the square of AC, together with the rectangle under DE and CB, that is, the rectangle under AC and CB. This, &c.

ART. 92. *If a right line be divided into any two parts, the square of the whole line is equal to the sum of the squares of the parts together with twice the rectangle under the parts.*

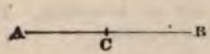
Let AB be a right line divided at C. Then the square of AB is equal



to the square of AC and the square of CB, together with twice the rectangle under AC and CB.

DEM. By ART. 90, the square of AB is equal to the rectangle under AB and AC, together with the rectangle under AB and CB; but by ART. 91, the rectangle under AB and AC is equal to the square of AC, and the rectangle under AC and CB; and by the same ART. the rectangle under AB and CB is equal to the square of CB and the rectangle under AC and CB. Hence the square of AB is equal to the square of AC, and the square of CB, and twice the rectangle under AC and CB. This, &c.

ART. 93. *The square of a right line is equal to four times the square of its half.*



Let AB be a right line divided *equally* at C. Then the square of AB is equal to four times the square of AC.

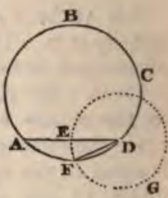
DEM. By ART. preceding, the square of AB is equal to the square of AC, and the square of CB, together with twice the rectangle under AC and CB. But the square of CB is equal to the square of AC, by ART. 33; and the rectangle under AC and CB is also equal to the square of AC, by DEF. XIII. Hence the square of AB is equal to four times the square of AC. This, &c.

NOTES TO PART II.

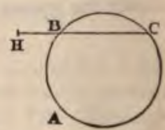
Note R. These definitions and the Observation, occurred in the problems of Part I., being necessary for the constructions there effected. They are here repeated, in order to render the theoretical and practical parts of our work distinct and complete.

Note S. This is a theorem, and should not be assumed, as it is in Euclid's definition of a circle, viz.: "A circle is a plane figure bounded by one line, such that all right lines drawn from a certain point *within the figure* to the circumference, are equal to one another." No one part of a definition should include another, else the latter would be superfluous.

Let ABC be a circle. The centre falls within the circumference.



Dem. It cannot fall on the circumference, as at D. For draw any line DA meeting the circle again at A, and with the point D as centre, and DE a *part* of DA, as radius, conceive the circle EFG described meeting the circle ABC in the point F. Then DF is equal to DE, and consequently less than DA. Hence, by Def. XV., D is not the centre of the circle ABC.

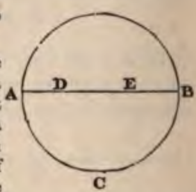


Again: the centre cannot fall without the circumference, as at H. For draw any line, HC, meeting the circle in two points, B and C, and it is plain that HB, HC are not equal. Hence, H is not the centre of the circle ABC, by Def. XV.

As the centre falls neither on, nor without the circumference, it must consequently fall within. This, &c.

Note T. The principle demonstrated in this article is entirely omitted in the common Euclids.

Let ABC be a circle. It cannot have two centres as at D and E.



Dem. For *suppose* the points D and E to be both centres of the circle ABC. Join DE by a right line, and produce it both ways to meet the circumference in A and B. Now, if D be a centre, DA equals DB, and therefore EA (which is greater than DA) is greater than DB. But if E be also a centre, EA equals EB, and therefore is less than DB. Consequently, *on this supposition*, EA would be at the same time greater and less than DB, which is impossible; therefore this supposition is false. Hence D and E are

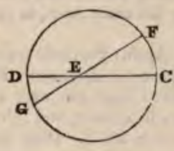
not both centres; nor any other two points, as may be shown in the same manner. This, &c.

NOTE U. A Secant is often confounded with a *chord;* but, properly speaking, the latter is terminated both ways in the circle, whilst the former is not.

NOTE V. The definition of a *Tangent* given in Euclid is vague and unsatisfactory, viz: "A right line is said to touch a circle, when it meets the circle, and being produced does not cut it." Here the meaning of the word "cut" is just as obscure as that of the word "touch," which it is brought to explain. We want a mathematical distinction between them, both being species of *meeting.*

LEGENDRE also, in his GEOMETRY, commits an oversight in defining a tangent and secant as we have done, without previously showing 1°. that a right line *can* meet a circle in but two points, 2°. that a right line *may* meet a circle in but one point.

NOTE W. This is a direct proof; that given in Euclid is an indirect one. The theorem is also improperly enounced in Euclid, for the condition of the angles, "opening towards the same hand," is omitted; which, if the point be within the circle, is indispensable. For in the opposite circle, though the angles FEC, GED, may be equal, yet the lines EF, EG, are not equal.



NOTE X. The demonstration of this theorem is different from that given in Euclid. It is chosen, 1°. because the demonstration given in Euclid is objectionable, assuming a theorem of BOOK Vth, which has not yet been proved, viz: Prop. XX., Book III., whose proof in Euclid depends on Book V.; 2°. because the two next theorems follow from this at once, without being split into cases, as is necessary in the other method; 3°. because it is curious to see how a result apparently so remote can be derived from the mere definition of a circle.

NOTE Y. Simson, in his edition of Euclid, divides his proof of this theorem into cases,—but leaves out one altogether, scil. where a side of the angle passes through the centre of a circle.

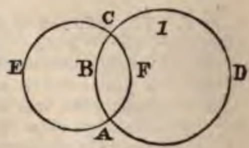
The proof by cases usually given is not considered legitimate, assuming the theorem of BOOK V. mentioned above; we have therefore substituted another proof, which is at once general and legitimate.

NOTE Z. This theorem follows very prettily, and without construction, from preceding Article: merely by considering that the angle at the centre is now equal to two right angles.

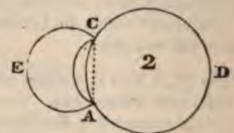
NOTE AA. Let ABCD, ABCE, be two different circles. They cannot meet in three points, such as A, B, and C.

DEM. *Suppose* them to meet in three points, A, B, and C; and draw the right lines AF, BF, CF, to the centre F of either circle ABCE. Then these three right lines are equal. But by ART. 76, the point F is not the centre of the circle ABCD; and therefore,

by Art. 68, these three lines would *not* be all equal. Hence the above supposition is false; that is, the circles do not meet in three points. Which, &c.



Note BB. Let ACD be any circle, and ACE another, meeting it in two points A and C. Then, one portion of ACE, as ABCE will be wholly without ACD, and the other portion of ACE, as ABCF, will be wholly within it; as in fig. 1.



Dem. The circle AEC cannot be wholly without the circle ACD, as in fig. 2; for then the chord AC being wholly within the circle ACD, by Art. 54,—would be wholly without the circle AEC. Which, by the same Art., is impossible.

Again: the circle ACE cannot be wholly within the circle ACD. For let it be *supposed* that AEC were a circle meeting ACD in A and C, and lying wholly within it as in fig. 3. Draw the common chord AC, and the perpendicular right line BD through its middle point, meeting both circles at I and B, between the points A and C. This line BD, by Art. 50, passes through the centre H and G of both circles. Let either of the points A or C be joined with both centres. Then in the triangle AHG, the sides GH and HA are together greater than GA, by Art. 43; consequently they are together greater than GB, as G is the centre of the circle ABD; therefore GH, together with HI, (which, as H is the centre of the circle ACE, is equal to HA,) are greater than GB; that is, on the *supposition* that the circle ACE were wholly within the circle ACD, the part GI would be greater than the whole GB, which is impossible. Therefore, that supposition is false; the circle AEC does not fall wholly within the circle ACB.

Hence, as the circle AEC falls neither wholly within, nor wholly without the circle ACD, one portion of it must fall within, and the other without the circle ACD. Which, &c.

Note CC. The definition of touching circles given in Euclid is defective in the same way as that of the Tangent, viz., "Circles are said to touch one another, which meet, but do not cut one another." Besides, the definition adopted in our treatise saves the necessity of introducing a theorem to prove that circles cannot touch in more points than one, which makes Prop. XIII., Book III., of Euclid. It may be observed, that the proof given in Simson's Euclid of the said Proposition is erroneous, though he substitutes it as a better one than that given by Euclid himself.

NOTE DD. Euclid defines at once "equal circles to be those whose diameters are equal," which is not properly a definition, but a theorem.

NOTE EE. By this demonstration we save the necessity of anticipating a definition which rightly belongs to the Doctrine of Proportion; and of introducing two theorems, the XXIIIrd and XXIVth of Euclid, BOOK III.

NOTE FF. Euclid's proof of this theorem is indirect, a method which, if possible, is always better avoided.

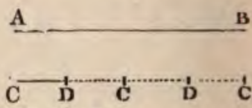
NOTE GG. These latter Articles are properly derivable from the Doctrine of Proportion, developed in BOOK VIth of Euclid, and in PART III. of this treatise. But as the reader may not choose to enter into that doctrine, it was thought prudent to introduce these Articles here, in order that the doctrine of the circle might be complete, and furnish him with all the results necessary to practice without any further reading.

PART III.

Of Ratios, or Proportion.

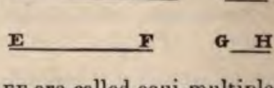
LESSON XI.

DEF. XXXII. When one magnitude exactly equals another magnitude added to itself any integral * number of times, the greater magnitude is called a *Multiple* of the lesser; and the lesser a *Submultiple* of the greater. [See NOTE HH.]



Thus, if AB equals CD added to itself any integral number of times (suppose 4), *without a remainder*, AB is called a multiple of CD, and CD a submultiple of AB.

DEF. XXXIII. When two multiples contain their respective submultiples exactly an equal number of times, they are called *Equi-multiples*. And when two submultiples are contained in their respective multiples exactly an equal number of times, they are called *Equi-submultiples*.



Thus, let AB, EF, be respectively multiples of CD, GH. If AB contains CD exactly as often as EF contains GH, neither more nor less, then AB and EF are called equi-multiples of CD and GH. Also, CD and GH, are called equi-submultiples of AB and EF.

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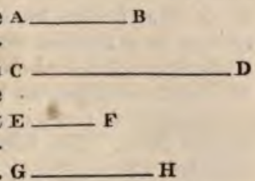
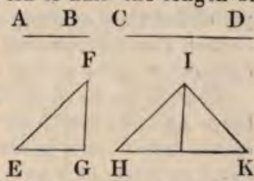
* An integral number is a *whole* number, such as 1, 2, 3, 4, &c. It is opposed to a fractional or broken number $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, &c.

DEF. XXXIV. *Ratio* is the mutual relation of two quantities of the same kind, to each other, with respect to magnitude. [See NOTE II.]

Obs. All quantities of the same kind have a certain ratio, or proportion to one another. Thus, in the annexed plate, the right line CD is twice as long as the right line AB; that is, the ratio or proportion of CD to AB is 2 to 1. Also, if the right line GH be three times as long as the right line EF, then the ratio or proportion of GH to EF will be 3 to 1, &c.

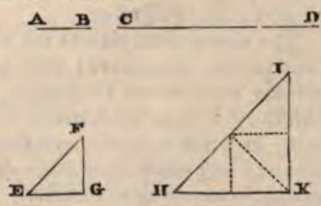


But two quantities may have the same ratio to each other as two other quantities have. Thus, if there be two right lines AB, CD, one of which AB is half the length of the other CD, and if there be also two triangles EFG, HIK, one of which EFG is half the size of the other HIK,—then the lines AB, CD, have to each other the same ratio as the triangles EFG, HIK, have to each other.

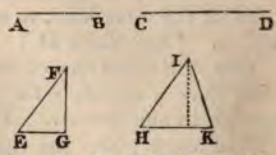


In the same manner, if one quantity be contained in a second any integral number of times exactly, and if a third quantity be contained in a fourth an *equal* number of times exactly,—then the first quantity has the same ratio to the second that the third has to the fourth.

Thus, if the right line AB be contained, say 4 times exactly in CD, and if the triangle EFG be also contained 4 times exactly in HIK, then AB has evidently the same ratio to CD that EFG has to HIK.

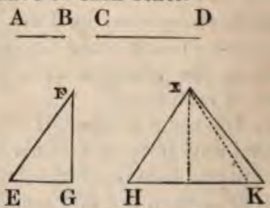


But although AB may not be contained in CD any integral number of times *exactly*, these quantities AB, CD, nevertheless have a certain ratio to each other; and two other quantities may have the same ratio to each other.



Thus, if CD be once and a half as long as AB, and HIK once and a half as big as EFG,—then CD has the same ratio to AB as HIK has to EFG. Thus although the *whole* line AB is not contained any integral number of times exactly in CD, yet its *half* is,—videlicet, 3 times. The half the triangle EFG is also contained exactly 3 times in HIK; hence the line AB must have the same ratio to CD that the triangle EFG has to HIK.

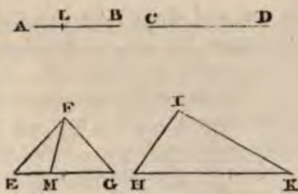
In this manner we are able to judge if the ratio of any two quantities, such as the above, to each other, be the same as that of any two other quantities to each other. Namely, if a submultiple of the first quantity be contained exactly as often in the second quantity as an *equi*-submultiple of the third quantity is contained exactly in the fourth, —then the first pair of quantities have the same ratio to each other as the second pair have to each other.



Thus, if CD be equal to twice AB and a fourth part of AB, and if HIK be also equal to twice EFG and a fourth part of EFG,—then CD has to AB the same ratio that HIK has to EFG. Because CD contains the fourth part of AB 9 times, and HIK contains the fourth part of EFG exactly as often.

The above observations are not intended to be demonstrative, but illustrative; they merely declare the *popular* notions of Ratio or Proportion, not the *mathematical* definition. But as the latter is founded upon the former,—being in truth only the same idea rendered a little more accurate and comprehensive,—it will be so much the easier to step from one to the other.

DEF. XXXV. Two quantities are said to have the same ratio to each other as two other quantities have, when every submultiple of the first quantity is contained in the second the same integral number of times that an *equi*-submultiple of the third quantity is contained in the fourth.

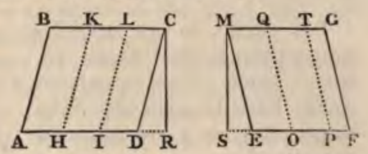


Thus let AB, CD, be two right lines, and EFG, HIK, two triangles; also, let AL be any submultiple of AB, and EFM an *equi*-submultiple of EFG: then,—if AL (and every other submultiple of AB) be contained in CD, as often as EFM (and every other *equi*-submultiple of EFG) is contained in HIK,—AB is said to have the same ratio to CD as EFG has to HIK. [See NOTE KK.]

Obs. When two quantities have the same ratio as two other quantities, the four quantities together are called four *proportionals*.

LESSON XII.

ART. 94. *Parallelograms which have equal altitudes have to each other the same ratio as their bases.*



Let AC, EG, be two parallelograms, having equal altitudes CR, MS. Then AC has the same ratio to EG as the base AD has to the base EF.

DEM. Divide AD into any number of equal parts *, AH,

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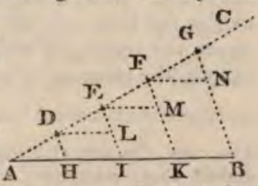
* PROB. XV. *To divide a given finite right line into any number of equal parts.*



Let AB be the given right line. It is required to divide it into a given number of equal parts.

CONS. Draw from the point A the right line AC, making an angle with AB. Take on the line AC the portions AD, DE, EF, FG, &c., all equal to each other, and as many in number as the parts into which AB is required to be divided. Join BG, and draw FK, EI, DH, parallel to BG. Then, AB will be divided at the points H, I, K, &c. into the required number of equal parts.

DEM. For, if we draw DL parallel to AB; then in the triangles

HI, ID, and draw the parallels HK, IL, forming the parallelograms AK, HL, IC, by DEF. XI. These parallelograms are all equal, by ART. 24. Consequently there being an equal portion of the parellelogram AC for every equal portion of the base AD,—AK and AH are respectively *equi*-submultiples of AC and AD. Take EO, OP, severally equal to AH, as often as possible on the base EF, and draw the parallels OQ, PT, forming the parallelograms EQ, OT, which are all equal by ART. 24, to each other. But these parallelograms are also equal to the parallelograms AK, HL, IC, by ART. 28; and therefore as often as AH is contained in EF, so often is AK contained in EG.—In the same way it may be shown that every other submultiple of AD is as often contained in EF as an equi-submultiple of AC is contained in EG.—Hence, by DEF. XXXV., AC has the same ratio to EG as AD has to EF. This, &c.

ART. 95. *Parallelograms which have equal bases, have to each other the same ratio as their altitudes.*

Let AC, EG, be two parallelograms having equal bases, AD, EH. Then, these parallelograms have to each other the same ratio as their altitudes DK, HM.

DEM. Draw AI, EL, parallel respectively to DK, HM, and produce KB, MF, to meet them. Then AK will be a rectangle equal to AC, and EM a rectangle equal to EG, by ART. 23. Now, if we take for the altitudes of these rectangles the lines AD, EH, respectively, as these are given equal, the rectangles will have to each other, by ART. preceding, the same ratio as their bases DK, HM;—hence the

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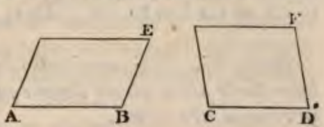
ADH, DEL, the side AD is equal to the side DE by CONSTRUCTION, and the angles DAH, ADH, are respectively equal to the angles EDL, DEL, by ART. 14.—Consequently, by ART. 46, the side AH is equal to the side DL. In the same way it is proved that EM, FN, &c., are respectively equal to AH. But DL, EM, FN, are respectively equal to HI, IK, KB. by ART. 20; hence AH, HI, IK, KB, are all equal to each other. This, &c.

parallelograms (which are equal to the rectangles) will also have the same ratio to each other as DK has to HM. This, &c.

DEF. XXXVI. When a certain part of one magnitude has the same ratio to a certain part of another magnitude as a second part *of this other* has to a second part of the first,—the proportion subsisting between these four parts is called, for brevity's sake, *reciprocal* proportion.

Thus, in the annexed figure, if one of the sides AB, belonging to the parallelogram AE, has the same ratio to one of the sides, CD, belonging to the parallelogram CF, as the adjacent side, DF, of *this* parallelogram, has to the adjacent side, BE, of the first,—then we say that these four sides AB and BE, CD and DF, are *reciprocally* proportional.



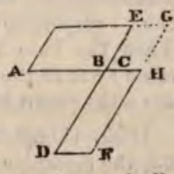
DEF. XXXVII. When two parallelograms have an angle in the one equal to an angle in the other, they are called *equi-angular* parallelograms.

It is obvious from ARTS. 21 and 13, that parallelograms which have *one* angle in each equal, have *all* the angles of the one respectively equal to all the angles of the other. It is likewise obvious that all rectangles are equi-angular to each other.

ART. 96. *Two equal parallelograms, which are also equi-angular, have the sides about their equal angles reciprocally proportional.*

Let AE, CF, be two equal parallelograms, having the angle ABE of the one equal to the angle DCH of the other.

Then AB has the same ratio to CH as CD has to BE.



DEM. Let the side CH be so placed that it may make one right line with AB, and that the equal angles ABE, DCH, may be vertically opposite. Since ABE and ECH are equal to two right angles by ART. 9, and since DCH is given equal to ABE, therefore also DCH and ECH are together equal to two right angles. Consequently, by ART. 10, CD and CE are in one right line.

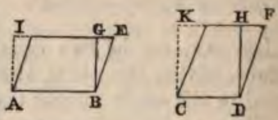
Draw EG, HG, respectively parallel to CH and CE; then CEGH will be a parallelogram, DEF. XI.—Now, by ART. 94, AB has the same ratio to CH as the parallelogram AE has to the parallelogram CG, or, which is the same thing, as the parallelogram CF has to the parallelogram CG. But CD has also the same ratio to BE as the parallelogram CF has to the parallelogram CG,—hence AB has the same ratio to CH as CD has to BE. This, &c.

ART. 97. *Two equi-angular parallelograms which have their sides about their equal angles reciprocally proportional are equal.*

Let AE, CF, in the above figure, be two parallelograms having the angle ABE in the one equal to the angle DCH in the other; and let the side AB have the same ratio to the side CH as the side CD has to the side BE,—then the parallelograms AE and CF are equal.

DEM. The above construction remaining, by ART. 94, AE has the same ratio to CG as AB has to CH, or, by the terms of the present ART., as CD has to BE. But CF has also to CG the same ratio as CD has to BE, by ART. 94.—Hence, AE and CF having both the same ratio to CG, are equal. This, &c.

ART. 98. *Equal parallelograms have their bases and altitudes reciprocally proportional.*



Let AE, CF, be two equal parallelograms, and BG, DH, their respective altitudes. Then the base AB has the same ratio to the base CD as the altitude DH has to the altitude BG.

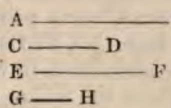
DEM. Complete the rectangles AIGB, CKHD. By ART. 23, the rectangle AG is equal to the parallelogram AE, and the rectangle CH to the parallelogram CF. Therefore the rectangles AG and CH are equal, because the parallelograms are so.—Hence, by ART. 96, the sides AB and BG, CD and DH are reciprocally proportional,—that is, AB has the same ratio to CD as DH has to BG. This, &c.

Art. 99. *Parallelograms which have their bases and altitudes reciprocally proportional are equal.*

In the above figure, let the given parallelograms be AE, CF; and let the base AB have the same ratio to the base CD, that the altitude DH has to the altitude BG. Then the parallelograms AE, CF, are equal.

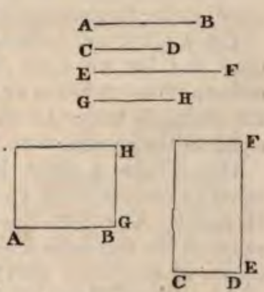
Dem. By Art. 97, the rectangles AG, CH, are equal, because the sides about their equal angles,—namely AB and BG, CD and DH,—are granted reciprocally proportional. But by Art. 23, these rectangles are respectively equal to the parallelograms AE, CF; hence therefore the parallelograms AE CF are also equal. This, &c.

Obs. If four right lines be proportionals, the two middle ones are called *Means*, and the two other *Extremes*. Thus if AB have the same ratio to CD as EF has to GH, the lines CD and EF are called *Means*, and the lines AB, GH, *Extremes*.



Art. 100. *If four right lines be proportionals, the rectangle under the extremes is equal to the rectangle under the means.*

Let AB, CD, EF, and GH, be four proportional right lines. Then the rectangle, AH, under AB and GH, is equal to the rectangle, CF, under CD and EF.



Dem. The parallelograms AH, CF, being rectangles, are equi-angular; and they have also their sides about the equal angles reciprocally proportional, viz.: AB has the same ratio to CD as EF has to GH, by the terms of the present Art.—Hence, by Art. 97, the rectangle AH is equal to the rectangle CF. This, &c. [See Note LL.]

Art. 101. *If there be four right lines, and the rectangle under any two of them equal to the rectangle under the remaining ones, these right lines are four proportionals.*

As in the above figure, let there be four right lines, AB, CD, EF, GH, and let the rectangle AH under any two of

them, as AB and GH, be equal to the rectangle CF under the remaining ones CD and EF. Then AB has the same ratio to CD as EF has to GH.

DEM. AH and CF are equal and equi-angular parallelograms; hence, by ART. 96, they have their sides about the equal angles reciprocally proportional,—that is, AB has the same ratio to CD as EF has to GH. This, &c.

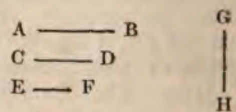
Obs. Hitherto we have considered only two quantities as having the same ratio to each other as two given quantities; but it is evident that two *other* quantities may also have the same ratio as the given quantities, or the second pair of quantities; and so on, the same ratio may hold between a fourth pair, a fifth, &c., &c.

DEF. XXXVIII. If there be a series of magnitudes of the same kind, in which the 1st has to the 2nd the same ratio as the 2d has to the 3d; and the 2d has to the 3d the same ratio as the 3d has to the 4th; and the 3d has to the 4th the same ratio as the 4th has to the 5th; and so on—then, these magnitudes are said to be in *continued* proportion.

Obs. It is evident that in order to institute a comparison of ratios, there must be at least *three* quantities.

ART. 102. *If three right lines be proportionals, the rectangle under the extremes is equal to the square of the mean.*

Let AB, CD, EF, be three right lines, and let AB have the same ratio to CD as CD has to EF. Then the rectangle under AB and EF is equal to the square of CD.



DEM. Take a right line GH equal to CD, and then AB is to CD as GH (or CD) is to EF*. Hence by ART. 100, the rectangle under AB and EF is equal to the rectangle under CD and GH,—*i. e.* equal to the square of CD (DEF. XIII). This, &c.

ART. 103. *If there be three right lines, and the rectangle under any two of them equal to the square of the third, these three right lines are proportionals.*

In the above figure, let the rectangle under AB and EF be equal to the square of CD. Then AB is to CD as CD to EF.

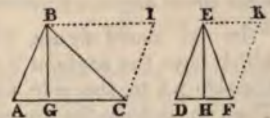
* This is the short way of saying that AB has the same ratio to CD, as GH has to EF. We shall use it henceforward.

DEM. Take GH equal to CD, and then the rectangle under AB and EF is equal to the rectangle under CD and GH, by terms of present ART.—Hence, by ART. 101, AB is to CD as GH is to EF,—*i. e.* as CD is to EF. This, &c.

LESSON XIII.

ART. 104. *Triangles which have equal altitudes are to each other as their bases.*

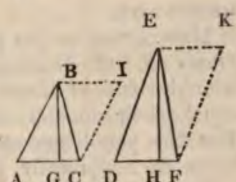
Let ABC, DEF, be two triangles, having their altitudes GB, HE, equal. Then ABC is to DEF as the base AC to the base DF.



DEM. By ART. 31, these triangles are halves of the respective parallelograms AI, DK, on the same bases, and with the same altitudes; consequently they have to each other the same ratio as their doubles, the parallelograms.—Hence, by ART. 94, the triangles are to each other as their bases. This, &c. [See NOTE MM.]

ART. 105. *Triangles which have equal bases are to each other as their altitudes.*

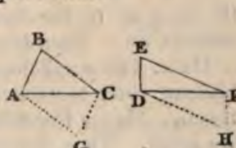
Let ABC, DEF, be two triangles, having their bases AC and DF equal. Then ABC is to DEF as the altitude GB to the altitude HE.



DEM. By ART. 95: because they are the halves of the parallelograms AI, DK. This, &c.

ART. 106. *Equal triangles which have also an angle in the one equal to an angle in the other, have the sides about these equal angles reciprocally proportional.*

Let ABC, DEF, be two equal triangles, which have also the angle at B equal to the angle at E. Then the side BA is to the side ED as the side EF to the side BC.



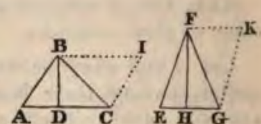
DEM. Complete the parallelograms ABCG, DEFH. By ART. 22, these parallelograms are equal, because they are doubles of the equal triangles ABC, DEF; and as they are also equi-angular, (DEF. XXXVII.,) the sides about their equal angles B and E are, by ART. 96, reciprocally proportional; that is, BA is to ED as EF is to BC. This, &c.

ART. 107. *Two triangles which have an angle in one equal to an angle in the other, and have also the sides about these equal angles reciprocally proportional, are equal.*

In the above figure, let ABC, DEF, be two triangles, which have the angle at B equal to the angle at E, and also the side BA to the side ED as the side EF to the side BC. Then these triangles are equal.

The parallelograms ABCG, DEFH, are equal, by ART. 97, being equi-angular, and having the sides about their equal angles reciprocally proportional.—Hence, their halves, the triangles, ABC, DEF, are also equal. This, &c.

ART. 108. *Equal triangles have their bases and altitudes reciprocally proportional.*



Let ABC, EFG, be two equal triangles. Then AC is to EG as HF is to DB.

DEM. Complete the parallelograms AI, EK. Inasmuch as the triangles ABC, EFG, are granted equal, their doubles, the parallelograms, are also equal. But the parallelograms being equal, their bases and altitudes are reciprocally proportional, by ART. 98; that is, AC is to EG as HF to DB. This, &c.

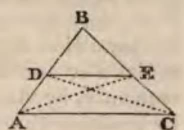
ART. 109. *Triangles which have their bases and altitudes reciprocally proportional are equal.*

In the above figure, let the two triangles ABC, EFG, have the base AC to the base EG as the altitude HF is to the altitude DB. Then the triangles will be equal.

DEM. The parallelograms AI, EK, are equal, by ART. 99, because their bases and altitudes are reciprocally proportional.—Hence the triangles also, which are their halves, are equal. This, &c.

LESSON XIV.

ART. 110. *If a right line be drawn parallel to any side of a triangle, and meeting the other sides, the segments of one of these sides have the same ratio as the segments of the other.*



Let ABC be a triangle, having the right line DE parallel to its side AC, and cutting AB, BC, respectively, into the segments DA, DB, and EC, EB. Then DA is to DB as EC to EB.

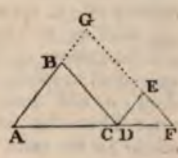
DEM. Join D and C, E and A. By ART. 104, the base DA is to the base DB as the triangle DEA to the triangle DEB. But by ART. 25, the triangles DEA and DEC are equal. Therefore DA is to DB as DEC to DEB; that is, by ART. 104, as EC is to EB. This, &c.

ART. 111. *If a right line meet the sides of a triangle, so that the segments of one side shall have the same ratio as the segments of the other,—and if the corresponding segments be at the same side of the intersecter,—this right line is parallel to the remaining side of the triangle.*

In a figure similar to the above, let DA have the same ratio to DB as EC has to EB. Then DE is parallel to AC.

DEM. By ART. 104, the triangle DEA is to the triangle DEB as DA to DB, that is, as EC to EB, by the terms of the present ART. But, by ART. 104, the triangle DEC is also to the triangle DEB as EC to EB.—Hence, the triangles DEA, DEC, both having the same ratio to DEB, are equal, and consequently, by ART. 27, DE is parallel to AC. This, &c. [NOTE NN.]

ART. 112. *If the angles of one triangle be respectively equal to the angles of another, the three sides of one triangle have to the corresponding sides of the other, respectively, the same ratio.*



Let ABC, DEF, be two triangles, having the angle ABC equal to the angle DEF, the angle ACB equal to the angle DFE, and consequently the angle BAC equal to the angle EDF. Then, as AC is to DF so is AB to DE;

and as AC is to DF so is BC to EF, and also as AB is to DE so is BC to EF.

DEM. Let the sides AC, DF, which are opposite equal angles, be placed so as to form one right line; and so that the triangles be at the same side of this line; and also that the equal angles BAC, EDF, may *open towards the same hand.* Now, since the angles ACB and DFE are equal, BC is, by ART. 16, parallel to EF, or to EF produced. Likewise, since the angles BAC and EDF are equal, AB is parallel to DE. Consequently, by ART. 18, AB, produced through B, will meet EF produced through E in some point, as G. Therefore, in

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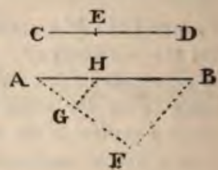
PROB. XVI. *To divide a right line into two parts which shall have the same ratio as the two parts of a given divided line.*

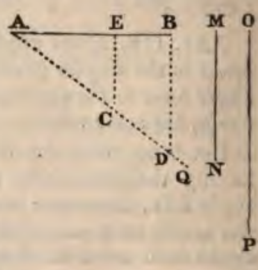


Let AB be the given right line, and CD the given divided right line. It is required to divide AB into two parts which shall have the same ratio to each other as CE to ED.

CONS. From either extremity of AB draw the right line AF making any angle with AB, and equal to CD; take AG equal to CE, which will leave GF equal to ED. Join BF, and draw GH parallel to BF. Then AB is divided at the point H as required.

DEM. By ART. 110, AH is to HB as AG to GF; that is, as CE to ED. This, &c.

PROB. XVII. *To cut off from a given right line a part which shall have a certain given ratio to the whole line.*



Let AB be the given right line. It is required to cut off from it a part which shall have a certain given ratio to the whole line

CONS. Let this given ratio be that of MN to OP. From either extremity of AB, draw the right line AQ, making any angle with AB; and take upon AQ, the portions AC, AD, equal respectively to MN, OP. Join DB, and draw CE parallel to DB. Then, the part AE has to the *whole* AB the ratio.

the triangle thus formed, AGF, the right line BC is parallel to GF, and the right line ED is parallel to GA. Conse-

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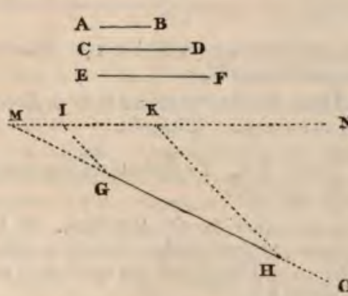
DEM. The two triangles AEC, ABD, having their angles respectively equal to one another, (ART. 14,) therefore, by ART. 112, AE is to AB as AC is to AD; that is, as MN to OP. This, &c.

PROB. XVIII. *To find a fourth proportional to three given finite right lines.*

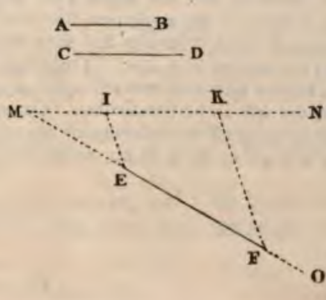


Let AB, CD, EF, be three given right lines. It is required to find a fourth such, that AB will have to CD the same ratio that EF has to the line found.

CONS. Draw two right lines MN, MO, making any angle, and take MI equal to AB, IK equal to CD, and MG equal to EF. Join IG, and draw KH parallel to IG. Then GH will be the fourth proportional sought.

DEM. IG being parallel to KH,—by ART. 110, MI is to IK as MG to GH; that is, AB is to CD as EF to GH. This, &c.

PROB. XIX. *To find a third proportional to two given finite right lines.*



Let AB, CD, be the two given right lines. It is required to find a third such, that AB will have to CD the same ratio that CD has to the line found.

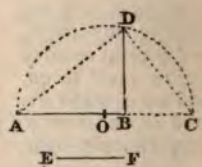
CONS. Draw two right lines MN, MO, making any angle, and take MI equal to AB, IK equal to CD, and also ME equal to CD. Join

quently BGED is a parallelogram, and therefore its opposite sides BG and DE, BC and GE, are equal. Hence, by ART. 110, BC being parallel to GF, AC is to DF as AB is to BG; that is, as AB is to DE. Hence also, by the same ART., ED being parallel to GA, AC is to DF as GE is to EF; that is, as BC is to EF. Finally, we have shown that AB is to DE as AC to DF, and that likewise BC is to EF as AC to DF; consequently AB is to DE as BC to DF. This, &c. [See NOTE OO.]

IE, and draw KF parallel to IE. Thus EF will be the third proportional sought.

DEM. By ART. 110, MI is to IK as ME to EF; that is, AB is to CD as CD to EF. This, &c.

PROB. XX. *To find a mean proportional between any two given finite right lines.*

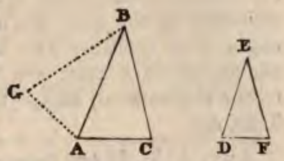


Let AB, EF, be the two lines. It is required to find another right line such, that AB will have to it the same ratio as it has to EF.

CONS. Produce AB until the produced part BC equals EF. Divide the whole AC equally at O, by PROB. V.; and from O as centre, and with OA as distance, describe the semicircle ADC. From the point B raise BD perpendicular to AC, by PROB. VII., and meeting the circumference at D. Then BD is the mean proportional sought.

DEM. Join DA, DC. In the triangles ADC, DBC, the angle at C is common, and the angle ADC, which is a right one, by ART. 72, is equal to the angle DBC, by ART. 8. Consequently the third angles of these triangles are also equal, by ART. 39; that is to say, the angles DAC, BDC. Therefore, in the triangles ABD, DBC, the angles DAB and BDC are equal, as also the angles ABD, DBC, because BD is perpendicular. Consequently, by ART. 39, the third angles of *these* triangles are equal. Hence, by ART. 112, AB is to BD as BD is to BC, or EF. This, &c.

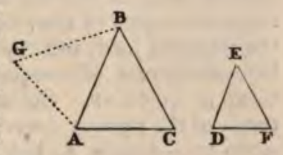
ART. 113. *If the three sides of any triangle have respectively to the three sides of another the same ratio, the angles of one triangle are respectively equal to the angles of the other.*



Let ABC, DEF, be two triangles having the side AB to the side DE as the side BC to the side EF, and also as the side AC to the side DF. Then the angle C is equal to F, A to D, and B to E.

DEM. At the extremities of any side of either triangle, as AB, let two angles BAG, ABG, be formed * equal respectively to the angles EDF, DEF, at the extremities of the corresponding side DE; then, by ART. 39, the remaining angles at G and F are also equal. Consequently, by ART. preceding, AB is to DE as BG to EF; but AB is to DE as BC to EF, by terms of present ART.; therefore BG is to EF as BC is to EF,—that is, BG and BC are equal. In the same manner it can be shown that AG and AC are equal.—Hence, by ART. 6, the three angles of the triangle ABC are respectively equal to the three angles of the triangle ABG,—that is, to the three angles of the triangle DEF, to which those of ABG were made equal. This, &c.

ART. 114. *If two sides of any triangle have respectively to two sides of another the same ratio, and likewise the angles contained by each pair of sides equal,—the other angles of the triangles will be also respectively equal.*



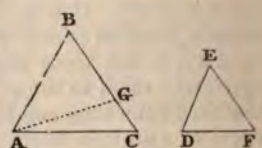
Let ABC, DEF, be two triangles having the side AB to the side DE as the side BC to the side EF; and also the angle ABC equal to the angle DEF. Then the angles BAC, BCA, are respectively equal to the angles EDF, EFD.

DEM. As in the last construction, make the angles ABG, GAB, respectively equal to DEF, EDF, and the angle at G

* By Problem VIII.

will be equal to that at F. Consequently, by ART. 112, AB is to DE as BG to EF; but AB is to DE as BC to EF, by the terms of the present ART.; therefore BG is to EF as BC to EF,—that is, BG and BC are equal.—Hence, as in the triangles ABC, ABG, the side AB is common, the side BC equal to the side BG, and the angle ABC equal to the angle ABG (because ABG has been made equal to DEF,) the angles BAC, BCA, are respectively equal, by ART. 2, to the angles BAG, BGA,—that is, to the angles EDF, EFD. This, &c.

ART. 115. *If two triangles have an angle in the one equal to an angle in the other; and if the sides containing a second angle in the former have, respectively, the same ratio to the sides containing a second angle in the latter; and if likewise the third angles of the triangles are either both acute or both obtuse, or both right; all the angles of these triangles are respectively equal to each other.*



Let ABC, DEF, be two triangles, having the angle B equal to the angle E, the side BA to the side ED as the side AC to the side DF, and the angles C and F, either both acute, both obtuse, or both right. Then, the angle C is equal to the angle F, and the angle A to the angle D.

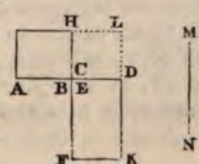
DEM. The angles BAC, EDF, are equal. For if we *suppose* one of them, as BAC, greater than the other, and that from the point A the line AG is drawn making with BA an angle ABG equal to EDF, and meeting BC in G. Then, in the triangles BAG, DEF, by ART 39, the angle BGA would be equal to EFD. Consequently, by ART. 112, BA would be to ED as AG to DF. But BA is to ED as AC to DF, by the terms of the present ART.; therefore AG would be to DF as AC to DF,—that is, AG and AC would be equal. By ART. 4, therefore, the angles ACG and AGC would be equal, and both of them acute by ART. 38. But if AGC be acute, AGB must be obtuse, and therefore also the angle EFD must be obtuse, because *on the above supposition* it is equal to AGB. The angles EFD and ACG are, however, both of the same

kind, and therefore, as ACG, *on the above supposition*, was proved acute, the angle EFD must be also acute. On the above supposition, therefore, the angle EFD would be obtuse and acute at the same time, which is impossible. Consequently, that supposition is false; the angle BAC is not greater than EDF; and in like manner it is demonstrable that the angle EDF is not greater than BAC.—Hence the angles BAC and EDF being equal, and the angles at B and E being granted equal, the remaining angles are also equal, by ART. 59. This, &c. [NOTE PP.]

Obs. In the last three Articles it is evident, as well from their demonstrations, as from ART. 112, that the sides opposite the equal angles in each triangle have, respectively, the same ratio to each other.

LESSON XV.

Obs. When we cannot compare two magnitudes as to size directly, we may often compare them by assuming a third magnitude; and by comparing both the former magnitudes with it, we are enabled to arrive at their ratio to each other. For instance, if each of them have the same proportion to the assumed magnitude, they are necessarily equal. But if they should not have the same proportion to the assumed magnitude, we may, nevertheless, discover their ratio by its means in the following manner. If we can find two right lines which have to each other the same ratio that one of the given magnitudes has to the assumed magnitude, and if moreover we can find a third right line which has to the *second* right line the same ratio as the assumed magnitude has to the other given magnitude, it is evident that the first right line will have to this third the same ratio that the first given magnitude has to the other given magnitude. And thus the ratio of these two lines indicates the ratio of the two given magnitudes.



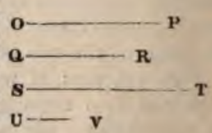
For example: If we wish to compare the two rectangles AH, CK, as to magnitude. Let them be placed as in ART. 96, and complete the rectangle CL. Then comparing AH with CL, we know by ART. 94, that these rectangles have the same ratio

to each other as AB has to CD. And comparing CL with CK, we know that these rectangles have the same ratio to each other as HC has to EF. Now if we can find a line MN (which is easily done,) such that CD may have to MN the same ratio as HC has to EF, then CL will be to CK as CD to MN. Hence, because AH is to CL as AB to CD; and CL is to CK as CD to MN; it follows that AH is to CK as AB to MN. These two right lines AB, MN, indicate, therefore, the ratio of these two rectangles AH, EK.

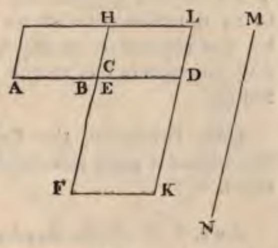
It is evident in the above operation that, in calculating by such means the ratio of the two given rectangles to each other, we combine, or compound, *two* ratios; for the ratio of AB to MN, which expresses the ratio of the given rectangles, is made out by combining the two ratios of AB to CD, and of CH to EF, which are respectively the ratios of one given rectangle to the assumed rectangle, and of the assumed rectangle to the other given rectangle. The ratio of the given rectangles, therefore, involves, or (as geometricians express it) is *compounded* of, two ratios. And in the same way, if any two magnitudes are compared by the means of one or more intermediate ones, the ratio of the given magnitude to each other, involves two or more ratios, and is said to be *compounded* of them.

When, therefore, geometricians speak of a ratio compounded of several ratios, they mean only—that if a series of like magnitudes (generally right lines) be found, which have to each other, the 1st to the 2nd, the 2nd to the 3d, the 3d to the 4th, &c., these several ratios respectively,—then the ratio which they speak of is the same as the ratio of the first term in the above series to the last. It is merely a form of speech to save circumlocution.

Thus, when we say that A has to B a ratio compounded of the ratios of a to b, c to d, and e to f, we mean only,—that if the line OP be to QR as a is to b, and the line QR to ST as c to d, and the line ST to UV, as e to f,—then A has the same ratio to B as the line OP to the line UV.



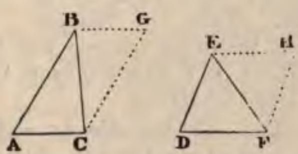
ART. 116. *Equi-angular parallelograms have to each other the ratio compounded of the ratios of the sides about the equal angles.*



Let AH, EK, be two equi-angular parallelograms. Then AH has to EK a ratio compounded of the ratio of the side AB to the side CD, and the ratio of the side BH to the side EF.

DEM. Take a right line MN, such that CD will have to MN the same ratio as BH has to EF. By ART. 94, AH is to CL as AB to CD; and CL is to EK (as BH to EF, or) as CD to MN.—Hence AH is to EK as AB to MN; but AB has to MN the ratio compounded of the ratios of AB to CD and of BH to EF, therefore, also, AH has to EK a ratio compounded of the same ratios. This, &c.

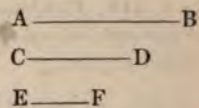
ART. 117. *Triangles which have an angle in the one equal to an angle in the other have to one another a ratio compounded of the ratios of the sides about the equal angles.*



Let ABC, DEF be two triangles, having the angle BAC equal to the angle EDF. Then these triangles have to each other a ratio compounded of the ratios of AB to DE, and of AC to DF.

DEM. Complete the parallelograms AG, DH. By ART. preceding, AG has to DH a ratio compounded of the ratios of AB to DE, and of AC to DF; hence also their halves, the triangles, have the same ratio. This, &c. [NOTE QQ.]

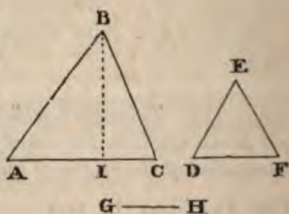
DEF. XXXIX. When three magnitudes are proportionals, the first is said to have to the third a ratio *duplicate* of that which it has to the second.



Thus, if AB be to CD as CD to EF, then the ratio of AB to EF is said to be duplicate of the ratio of AB to CD; because CD having to EF the same ratio as AB to CD, by the repetition or *duplication* of this ratio, which the first magnitude has to the second, the third magnitude is found.

Obs. Inversely, the first magnitude is said to have to the second a ratio *sub-duplicate* of that which it has to the third.

ART. 118. *If the angles of one triangle be respectively equal to those of another, these triangles have to each other a ratio duplicate of that which their corresponding sides have to each other.*



Let ABC, DEF, be two triangles having the angles, A, B, and C, respectively equal to those at D, E, and F. Also let GH be a third proportional to AC and DF*; that is, let AC be to DF as DF to GH. Then the triangle ABC has to the triangle DEF the same ratio as AC has to GH,—*i. e.* a ratio *duplicate* of that which AC has to DF (DEF. XXXIX.)

DEM. On the side AC, and from one of its extremities, C, take CI equal to GH. Draw BI. Now, as BC is to EF as AC to DF, by ART. 112,—therefore BC is to EF as DF to GH, or CI. Consequently, the triangles BCI, DEF, are equal, by ART. 107, having the sides about their equal angles reciprocally proportional. Also, by ART. 104, the triangle ABC is to the triangle BCI as AC to CI; and hence the triangle ABC is to the triangle DEF as AC to CI, or as AC to GH,—that is, in the *duplicate* ratio of AC to DF. Which, &c.

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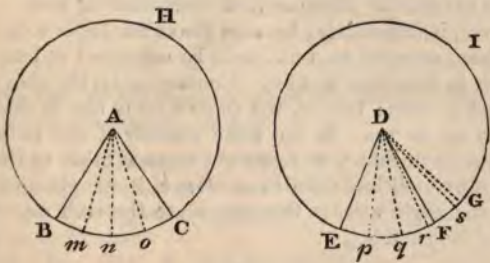
* By PROB. XIX.

ART. 119. *In equal circles, angles at the centres have to each other the same ratio as the arches on which they stand.*



In the equal circles BCH, EGI, let BAC, EDF, be respectively angles at the centres. Then the angle BAC is to the angle EDF as the arch BC to the arch EF.

DEM. For *suppose* the angle BAC to be to the angle EDF, *not* as the arch BC to the arch EF, but as the arch BC to an arch EG, greater than EF. Divide, then, the angle BAC into any number of equal parts, BA*m*, *m*A*n*, *n*A*o*, *o*AC, such that they will be severally less than, or equal to FDG*. Also, take the angles ED*p*, *p*D*q*, *q*D*r*, *r*D*s*, respectively equal to BA*m*†. Now, by ART. 86, as the angles BA*m*, *m*A*n*, *n*A*o*, *o*AC, are equal, the arches on which they stand are equal, namely, B*m*, *mn*, *no*, *o*C. Therefore, whatever submultiple the angle BA*m* is of BAC, such a submultiple is the arch B*m* of BC. Likewise, as the angles ED*p*, *p*D*q*, *q*D*r*, *r*D*s*, are equal to each other, the arches E*p*, *pq*, q*r*, *rs*, are equal to each other, by the same ART.; and as these angles are equal to the smaller angles in the circle BCH, so these arches are equal to the smaller arches in that circle. Consequently, as often as the submultiple BA*m* is contained in EDG, so often is the submultiple B*m* contained in EG. But, *on the supposition above-mentioned*, the submultiple BA*m* would be contained in EDF as often as the submultiple B*m*

* By PROB. IV. We can continue the subdivision of an angle by that problem, *ad libitum*; and therefore it is evident can subdivide the angle BAC into parts smaller than the angle FDG, however small it may be.

† By PROB. VIII.

is contained in EG, by DEF. XXXV., because the angle BAC is *supposed* to be to the angle EDF, as the arch BC to the arch EG; therefore the submultiple BAm would be contained in EDF as often as it is contained in EDG. This, however, is impossible; because the submultiple BAm being less than, or equal to, FDG, must be contained at least *once* oftener in EDG than in EDF. Consequently, the above supposition is false; that is, the angle BAC is not to the angle EDF as BC to EG. In the same manner it can be proved that the angles BAC, EDF, are not to each other as the arch BC to any arch less than EF.—Hence, the angle BAC must be to the angle EDF, as the arch BC to the arch EF. This, &c. [NOTE RR.]

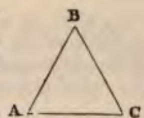
ART. 120. *In equal circles, angles at the circumferences have to each other the same ratio as the arches on which they stand.*

By ART. 71; because their doubles, the angles at the centre, have to each other that ratio, by preceding ART.

END OF THE ELEMENTS OF GEOMETRY

DEDUCTIONS AND USEFUL RESULTS FROM THE ELEMENTS OF GEOMETRY.

ART. 121. *Every triangle which has its three sides equal has also its three angles equal.*



Let ABC be a triangle of which the *sides* AB, BC, CA, are all equal to each other. Then also the angles at A, B, and C, are all equal to each other.

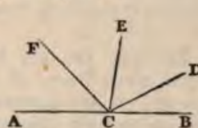
DEM. By ART. 4; because whichever pair of sides we take, the angles opposite them are equal.

ART. 122. *Every triangle which has its three angles equal has also its three sides equal.*

In the above figure let the *angles* at A, B, C, be all equal. Then also the sides AB, BC, CA, are all equal.

DEM. By ART. 5.

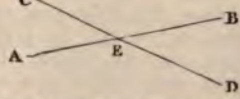
ART. 123. *Where several right lines meeting another right line at the same point, make angles with it, these angles are altogether equal to two right angles.*



Let the right lines CD, CE, CF, meet AB, in the same point, C. Then the angles ACF, FCE, ECD, DCB, are together equal to two right angles.

DEM. Proved in the same manner as ART. 9.

ART. 124. *Two right lines intersecting each other, make angles which taken together are equal to four right angles.*

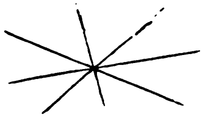


Let the right lines AB and CD intersect. Then the angles AEC, CEB, BED, DEA, are all together equal to four right angles.

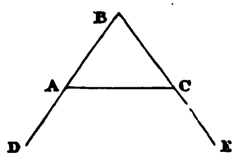
Dem. By ART. 9; for the angles AEC, CEB, are equal to *two* right angles, and the angles, AED, DEB, to two also.

ART. 125. *If several right lines intersect one another in the same point, all the angles taken together are equal to four right angles.*

This is evident [NOTE SS].



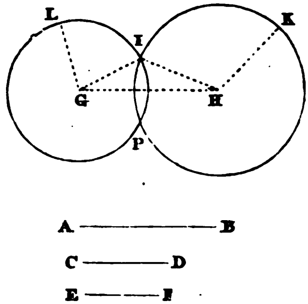
ART. 126 *A triangle which has two of its sides equal, if these equal sides be produced, will have the angles beneath the third side equal to each other.*



Let ABC be a triangle having its sides AB and BC equal. Let these sides be produced to D and E. Then the angles DAC and ECA are equal.

DEM. By ART. 9, the angles DAC and CAB are together equal to two right angles; also the angles ECA and ACB

PROB. XXI. *Given three finite lines, of which any two are greater than the third, to construct a triangle of which the sides shall be respectively equal to the given lines.*



Let AB, CD, EF, be the three given lines, of which AB and CD are greater than EF, AB and EF greater than CD, CD and EF greater than AB. It is required to construct a triangle, whose sides shall be respectively equal to AB, CD, and EF.

CONS. From any point G draw the right line GH, equal to one of the given lines AB, by PROB. II. From each extremity of GH draw the right lines HK, GL, respectively equal to CD and EF, by the same PROB. With the point G as centre, and with the distance GL describe the circle LIP; and with the point B

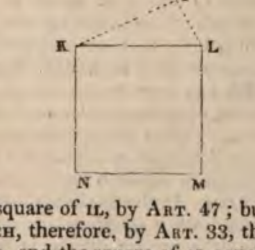
are together equal to two right angles. Hence, the angles DAC and CAB together are equal to ECA and ACB together. But by ART. 4, the angles CAB and ACB are equal; hence

as centre, and with the distance HK, describe the circle PIK. Join the points G and H with the point I, where these circles intersect. Then the triangle GIH is the triangle required.

DEM. GH is drawn equal to AB. GI is equal to GL, by DEF. 3, and therefore equal to EF. Also HI is equal to HK by the same DEF., and therefore equal to CD.—Hence the three sides of the triangle GIH are respectively equal to AB, CD, and EF.

Obs. In this manner a triangle may be constructed equal to a given one, by constructing a triangle whose three sides shall be equal respectively to the three sides of the given one. By ART. 7, these triangles are equal.

PROB. XXII. *To find a square equal to the sum of two squares.*



Let ABCD, EFGH, be the given squares. It is required to find a square equal to them both.

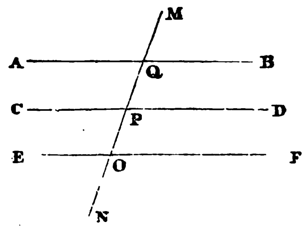
CONS. From any point I, draw the right line IK, equal to any side of either square, such as AD, by PROB. II. At either extremity of IK raise IL perpendicular to IK, by PROB. VII., and equal to any side of the other square, EH, by PROB. III. Join KL, and describe the square KLMN, by PROB. X. Then KLMN is equal to ABCD and EFGH together.

DEM. As the angle at I is right, the square of KL is equal the square of IK together with the square of IL, by ART. 47; but as IK equals AD, and as IL equals EH, therefore, by ART. 33, the square of IK equals the square of AD, and the square of IL equals the square of EH. Hence the square of KL is equal to the square of AD together with the square of EH; that is, KLMN is equal to ABCD and EFGH together.

the remaining angles DAC and AEC are equal. [NOTE TT.]

ART. 127. *If two right lines be parallel to the same right line, they are parallel to one another.*

Let AB and CD be right lines parallel to EF. Then AB and CD are parallel to one another.



DEM. Draw the right line MN intersecting the three given lines. By ART. 14, the angles MQA and MOE are equal; and

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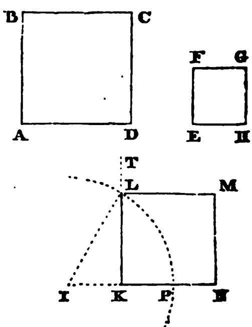
PROB. XXIII. *To find a square equal to the difference of two squares.*

Let ABCD, EFGH, be the given squares. It is required to find a square equal to the difference of these squares.

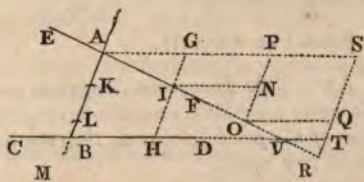


CONS. From any point I draw IP equal to AD, the side of the greater given square, by PROB. II.; and by PROB. III. cut off from IP a part IK equal to EH, the side of the lesser square. At the point K raise KT perpendicular to IP, by PROB. VII. Lastly, with I as a centre, and IP as distance, describe a circle cutting KT at L. Then the square KLMN, described upon KL, is equal to the difference between the square ABCD and the square EFGH.

DEM. As the angle IKL is right, therefore the square of IL is equal to the square of IK and the square of KL, by ART. 47; *i. e.* the square of KL is equal to the difference between the square of IL and the square of IK. But as IL equals IP, it equals AD; and IK equals EH.—Hence, the square of KL is equal to the difference between the square of AD and the square of EH; that is, KLMN is equal to the difference between ABCD and EFGH.

by the same ART., the angles MPC and MOE are equal. Hence the angles MQA and MPC are equal;—that is, by ART. 16, the lines AB and CD are parallel.

ART. 128. *If a right line intersect two right lines, and make the two internal angles at the same side of the intersecting line together less than two right angles, these two latter right lines will meet if produced sufficiently.*



Let AB be the right line intersecting two other right lines EF, CD, and making the two internal angles BAF, ABD, together less than two right angles. Then the lines EF and CD will meet, if produced sufficiently.

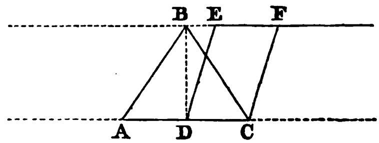
DEM. Draw AS parallel to BD; and since, by ART 13, the angles SAB, ABD, together are equal to two right angles, they must be greater than the angles FAB, ABD together. Therefore, taking away the common angle ABD from both, the angle SAB remains greater than the angle FAB, and therefore AF falls within AB and AS. Take any point G in AS, and draw GH parallel to AB, which will meet CD, by ART. 18, and therefore AF, as in the point I. Take also AK, KL, LM, successively equal to IG, until AM be greater than AB; and draw IN parallel to AG and equal to it. Take IO equal to AI, and join ON. Consequently the two triangles IAG, OIN, having the angles IAG and OIN equal, by ART. 14, and the containing sides also respectively equal, the side ON is equal to IG, and the angle AIG is equal to the angle ION. Therefore ON is parallel to IG, and therefore, by ART. 18, will meet AS, as in P, NP being equal to IG. In the same manner, taking OR equal to AI, drawing OQ parallel and equal to AG, and joining RQ, we may show that RQ is equal and parallel to IG, or ON. Therefore, if produced, it will meet AS, as in S, QS being equal to OP or AL. Consequently

SR is equal to AM, and being also parallel to it, BD will meet SR as in T, by ART. 18. But by the same ART., as BD is parallel to AS, ST is equal to AB, and therefore less than SR. Hence the point R of the line AF being below the line CD, while the point A of the same line is above CD, the line AF must intersect the line CD. This, &c. [See NOTE UU.]

ART. 129. *If a parallelogram and triangle be upon equal bases and between the same parallels, the parallelogram is double of the triangle.*

Proved as ART. 31.

ART. 130. *If a parallelogram and triangle be between the same parallels, and the base of the triangle double the base of the parallelogram, then the parallelogram and triangle are equal.*

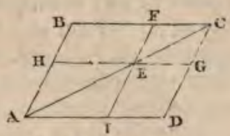


Let DEFC, ABC, be the parallelogram and triangle between the same parallels, and the base AC double of the base DC. Then DEFC and ABC are equal.

DEM. Draw BD. By ART. 25, the triangles ABD, DBC, are equal, and therefore the whole triangle ABC is double of the triangle DBC. But by ART. 31, the parallelogram DEFC is also double of the triangle DBC. Hence the triangle ABC and the parallelogram DEFC are equal.

DEF. XL. If through a given point in the diagonal of a parallelogram two right lines be drawn parallel respectively to two adjacent sides of the parallelogram, so as to make four parallelograms,—then, those two through which the diagonal runs, are called *parallelograms about the diagonal*, and the remaining two are called *complements* of the parallelograms about the diagonal.

Thus, if through the point E in the diagonal AC of the parallelogram ABCD, two right lines IF, HG, be drawn parallel respectively to the adjacent sides AB, BC, they form the parallelograms AHEI, EFCG, HBFE, EGDI, by DEF. XI. Then, AHEI and EFCG are called parallelograms about the diagonal AC, because AC runs through them; and HBFE, EGDI, are called complements of the parallelograms AHEI and EFCG.



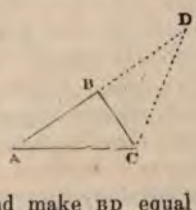
ART. 131. *In a given parallelogram the complements of the parallelograms about the diagonal are equal to each other.*

In the above figure the complements HBFE and EGDI are equal.

DEM. By ART. 22, the triangle ABC is equal to the triangle ADC, the triangle AHE to the triangle AIE, and the triangle EFC to the triangle EGC. Therefore taking away the triangles AHE, EFC, from the whole triangle ABC, there will remain the complement HBFE; and taking away the triangles AIE, EGC, from the whole triangle ADC (which is equal to ABC), there will remain the complement EGDI;—hence, these remaining complements are equal.

ART. 132. *If the square described upon one side of a triangle be equal to the squares described on the other sides of the triangle, taken together, the angle opposite to the first-mentioned side is a right angle.*

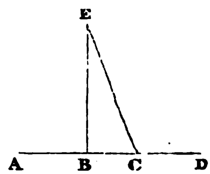
Let ABC be a triangle, the square of whose side AC is equal to the square of the side AB together with the square of the side BC. Then the angle ABC is a right angle.



DEM. At the point B raise BD perpendicular to BC, but at the other side of it from BA; and make BD equal BA. Join DC. By ART. 47, the square of CD is equal to the square of BD together with the square of BC; that is, the square of CD is equal to the square of AB together with the

square of BC; consequently, the square of CD is equal to the square of AC, and by ART. 34, the side CD is equal to the side AC.—Hence, the triangles ABC, DBC, having all their sides respectively equal, have likewise all their angles respectively equal, by ART. 6, and therefore the angle ABC equal to the angle DBC, that is, equal to a right angle.

ART. 133. *Two perpendiculars cannot be drawn from the same point to the same right line.*



Let EB, EC, be two right lines drawn from the same point E to the same right line AD. Then EB and EC cannot be *both* perpendicular to AB.

DEM. For if one of them, as EB, be perpendicular to AD, then by ART. 38, the angle EBC being a right angle, the angle ECB is acute. Hence EC is not perpendicular to AB, by DEF. VI.

ART. 134. *If two right lines be drawn from the same point to the same given right line, and if one of them be perpendicular, the other not, the perpendicular will fall at the side of the acute angle.*

In the above figure, let EB and EC be the right lines drawn from the same point E to the same right line AD, and let EB be perpendicular to AD, and EC not. Then EB falls at that side of EC where the latter forms an *acute* angle with AD.

DEM. By ART. 36, the angle ECA is less than the angle EBA, and therefore less than a right angle, as EB is perpendicular. Hence, EB falls at the side of the acute angle.

ART. 135. *If two right lines be drawn from the same point to the same right line, and if one of them be perpendicular, the other not, the perpendicular is less than the other.*

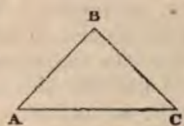
In the above figure EB is less than EC.

DEM. By ART. 38, as the angle EBC is right, the angle ECB is acute, and therefore less than EBC.—Hence by ART. 42, the side EC is greater than the side EB.

ART. 136. *Each angle of an equal-sided triangle is one-third of two right angles.*

DEM. By ART. 119, the angles are all equal; and as they are together equal to two right angles, by ART. 37, each alone must be equal to *one-third* of two right angles.

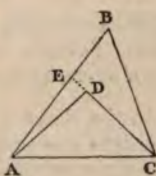
ART. 137. *In a right-angled triangle whose sides about the right angle are equal, the remaining angles are each equal to half a right angle.*



Let ABC be a triangle of which the angle at B is right, and the sides about it, BA and BC, equal. Then each of the angles BAC, BCA, is equal to half a right angle.

DEM. By ART. 37, the angles BAC and BCA are together equal to *one* right angle, the angle at B being right.—Hence, each of the angles BAC, BCA, is half a right angle, inasmuch as they are equal by ART. 4.

ART. 138. *If from a point within a triangle right lines be drawn to the extremities of any side, these are together less than the other two sides of the triangle, but contain a greater angle.*

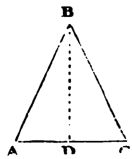


Let DA, DC, be drawn from the point D within the triangle ABC, to the extremities A and C of the side AC. Then DA and DC together are less than BA, BC, together; but the angle ADC is greater than the angle ABC.

DEM. Produce CD till it meets AB in E. By ART. 43, EB and BC are together greater than EC; therefore adding EA to both sides, AB and BC together will be greater than AE and EC together. But by ART. 43, AE and ED together are greater than AD; therefore adding DC to both sides, AE and EC together are greater than AD and DC together.—Hence, as it was shown that AB and BC together were greater than AE and EC together, AB and BC together must be still greater than AD and DC together.

Again: the angle ADC is greater than the angle AEC, by ART. 36, and by the same ART. the angle AEC is greater than the angle ABC.—Hence, the angle ADC must be still greater than the angle ABC.

ART. 139. *In a triangle which has two equal sides, the right line drawn from the vertex of the angle between them, perpendicular to the third side, divides that angle, and also the third side into two equal parts.*



Let ABC be a triangle having the sides BA, BC, equal. Then if BD be drawn perpendicular to AC, it divides the angle ABC into the equal parts ABD and CBD, and also the third side AC into two equal parts, AD and DC.

DEM. In the triangle ABD, CBD, the sides AB, CB are equal, the angle at A is equal to the angle at C (by ART. 4), and the right angles at D are also equal, by ART. 8.—Hence, by ART. 46, these triangles are in every respect equal to one another; that is, the angle ABD is equal to the angle CBD, and the side AD to the side DC.

ART. 140. *In a triangle which has two equal sides, a right line dividing the angle between them into two equal parts, if drawn to the third side, will divide it into two equal parts, and also be perpendicular to it.*

In the above figure, if BD be drawn dividing the angle ABC into two equal parts, ABD, CBD, and meeting the third side AC at D; then, the parts AD and DC are equal, and the line BD is perpendicular to AC.

DEM. The first part is evident from ART. 1; and the second from ART. 2 and DEF. VI.

ART. 141. *In a triangle which has two equal sides, a right line drawn from the vertex of the angle between them to the middle point of the third side, divides the opposite angle into two equal parts, and is also perpendicular to the third side.*

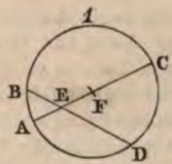
In the above figure, if BD be drawn to the middle point

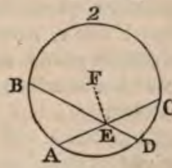
of AC; then, BD divides the angle ABC into two equal parts, ABD and CBD, and is also perpendicular to AC.

DEM. The first part is evident from ART. 6; and the second from ART. 6 and DEF. VI.

ART. 142. *In a circle, if two chords intersect, which are not both diameters, they do not divide each other into equal parts.*

In the circle ABCD, let the two chords AC, BD, intersect at the point E, which is not the centre. Then, AC, BD, are not *both* divided equally at E.





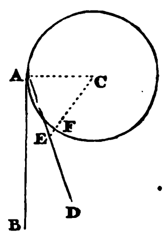
DEM. If one of these lines, as AC, pass through the centre F, the other does not (as in fig. 1), and therefore cannot pass through the middle point of the former, which *is* the centre. If neither pass through the centre, draw EF from their point of intersection to the centre. Then, let either of them, as AC, be divided equally at E, and the line EF will be perpendicular to AC, by ART. 52; consequently the angle FEA is a right one. Now, *suppose* the other BD also divided equally at E, the line EF would be also perpendicular to BD, by ART. 52, and the angle FEB consequently a right one. Therefore the angles FEA and FEB would be equal, by ART. 8, the part to the whole, which is impossible.—Hence, the above supposition is false; BD is not divided equally by AC. In the same manner we can show that if BD be equally divided at E, then AC is not.

ART. 143. *If from a point within a circle there can be drawn more than two equal right lines to the circumference, this point is the centre of the circle.*

By ART. 68.

ART. 144. *If a right line be drawn from the point of contact nearer the centre than the tangent, it cuts the circle.*

Let the right line AD be drawn from the point of contact, A, nearer the centre C, than the tangent AB. Then AD cuts the circle.



DEM. For *suppose* AD not to cut the circle, but to be wholly outside of it. Since the angle BAC is a right one, the angle DAC is acute; and therefore, by ART, 134, if a perpendicular from C be drawn to AD, it will fall at the side of the acute angle DAC. Let CE be that perpendicular. Now, in the triangle ACE, the angle CAE is less than CEA, CE being a perpendicular, by ART. 38. Therefore, by ART. 42, CA is greater than CE; that is, *on the above supposition*, CF, which is equal to CA, would be also greater than CE,—a part greater than the whole, which is impossible.—Hence, the above supposition is false; that is, AD does not lie wholly outside the circle*.

ART. 145. *At the same point of the same circle but one right line can be drawn touching the circle.*

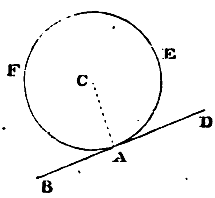
Evident from the preceding Article.

ART. 146. *If one side of a quadrilateral figure inscribed in a circle be produced, the external angle thus formed is equal to the internal remote angle of the quadrilateral.*

By ARTS. 69 and 9.

PROB. XXIV. *To draw a tangent at any given point in the circumference of a circle.*

Let A be any point in the circumference AEF. It is required to draw a right line, which shall be a tangent to the circle at the point A.



CONS. Join the point A with the centre C; and raise AD perpendicular to AC. Then AD is the required tangent.

DEM. By ART. 55 and DEF. XX.

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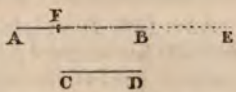
* Therefore, however small the angle DAB may be, or however great the angle DAC, the right line AD always cuts the circle, and no right line can be drawn to the point of contact *between* the tangent and the circumference.

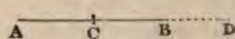
ART. 147. *The difference between the squares of any two unequal right lines is equal to the rectangle under the sum of the lines and their difference*



Let AB, CD, be any two right lines. Produce the greater of them, as AB to E, so that BE may be equal to CD: AE will be equal to the sum of AB and CD. From the point B take a portion BF on AB equal to CD, and AF will be equal to the *difference* of AB and CD. Then, the difference between the square of AB and the square of CD, is equal to the rectangle under AE and AF.

DEM. By ART. 90, the square of AB is equal to the rectangle under AB and FB, together with the rectangle under AB and AF; which former rectangle is, by ART. 91, equal to the square of FB (or of CD) and the rectangle under FB and AF. Therefore the square of AB is equal to the square of CD, the rectangle under FB and AF, and the rectangle under AB and AF; that is, the square of AB exceeds the square of CD, by the rectangle under FB and AF, together with the rectangle under AB and AF. But, as FB equals BE, the rectangle under FB and AF, together with the rectangle under AB and AF, is equal to the rectangle under BE and AF, together with the rectangle under AB and AF; and consequently to the rectangle under AE and AF, by ART. 89.—Hence, the square of AB exceeds the square of CD, by the rectangle under AE and AF; or in other words the difference between the square of AB and the square of CD is equal to the rectangle under the sum and the difference of AB and CD.

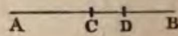
ART. 148. *If a right line be equally divided, and produced to any point,—then the rectangle under the whole line and the produced part, is equal to the difference between the square of half the original line, and the square of the line made up of that half and the produced part.*



Let AB be a right line divided equally at C, and produced to the point D. Then the rectangle under AD and DB is equal to the difference between the square of CB and the square of CD.

DEM. By ART. preceding, the difference between the square of CB and the square of CD, is equal to the rectangle under the sum of these lines and their difference; but, as AC equals CB, AD is their sum, and BD is their difference.

ART. 149. *If a right line be divided into two equal parts, and into two unequal parts, the rectangle under the unequal parts is equal to the difference between the square of half the line, and the square of the intermediate part.*



Let AB be a right line divided equally at C, and unequally at D. Then, the rectangle under AD and DB is equal to the difference between the square of AC and the square of CD.

DEM. By ART. 147, the difference between the square of AC and the square of CD is equal to the rectangle under the sum of these lines, and their difference; but AD is their sum, and (as AC equals CB) DB is their difference.—[NOTE VV.]

ART. 150. *If a right line be divided equally and unequally, the rectangle under the equal parts is greater than the rectangle under the unequal parts.*

By preceding ART.; for the rectangle under the equal parts is the square of half the line, which exceeds the rectangle under the unequal parts by the square of the intermediate part.

Obs. Hence, the more unequally a line is cut, the less will be the rectangle under the parts.

ART. 151. *If a right line be cut equally and unequally, the sum of the squares of the unequal parts is greater than the sum of the squares of the equal parts,—or, in other words, greater than twice the square of half the line.*

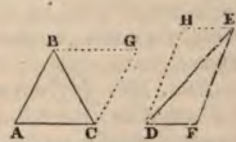
DEM. By ART. 92, the square of the whole line is equal to the sum of the squares of the unequal parts, together with twice the rectangle under those parts; and by ART. 93, the square of the whole line is equal to four times the square of half the line. Consequently, the sum of the squares of the unequal parts, together with twice the rectangle under these parts, is equal to four times the square of

half the line. But, inasmuch as, by the preceding ARTICLE, twice the rectangle under the unequal parts is less than *twice* the square of half the line,—hence the sum of the squares of the unequal parts must be greater than twice the square of half the line.

Obs. Also, it is plain from the above demonstration, that the more unequally the line is cut, the greater becomes the sum of the squares of the unequal parts.

ART. 152. *If two equal triangles have an angle in the one which together with the angle in the other is equal to two right angles, the sides about these angles are reciprocally proportional.*

Let ABC, DEF, be two equal triangles, and let the angle at A, together with the angle at F, be equal to two right angles. Then, the side AB is to the side EF as the side FD is to the side AC.



DEM. Complete the parallelograms AG and FH. By ART. 22, these parallelograms are equal, being doubles of equal triangles. They are also equi-angular, by DEF. XXXVII. —Hence, by ART. 96, AB is to DH (or EF) as DF is to AC.

ART. 153. *If two triangles have an angle in the one which together with an angle in the other is equal to two right angles, and if the sides about the given angles are reciprocally proportional,—then these two triangles are equal.*

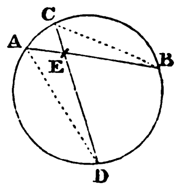
In the above figure, let the angle at A, together with that at F, be equal to two right angles, and let the side AB be to EF as DF to AC. Then the triangles ABC, DEF, are equal.

DEM. Complete the parallelogram AG and FH. These parallelograms are equi-angular, and the sides about their equal angles are granted reciprocally proportional, namely, AB is to HD (or EF) as DF to AC;—hence, by ART. 97, these parallelograms are equal, and therefore the triangles which are their halves are equal.

ART. 154. *If four right lines be proportionals, the parallelogram under the extremes is equal to an equi-angular parallelogram under the means.*

Proved as ART. 100.

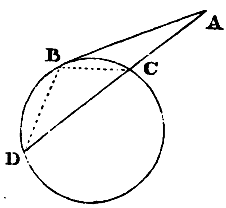
ART. 155. *If two chords of a circle intersect, the rectangle under the segments of one is equal to the rectangle under the segments of the other.*



Let the chords AB, CD, intersect at the point E. Then, the rectangle under AE and EB is equal to the rectangle under CE and ED.

DEM. Draw the lines AD and CB. By ART. 11, the vertical angles at E are equal; and, by ART. 70, the angles in the same segment DAB, DCB, are equal; and also the angles ADC, ABC.—Hence, as the triangles AED, CEB, are equi-angular to each other, AE is to CE as ED is to EB, by ART. 112; that is, by ART. 100, the rectangle under AE and EB is equal to the rectangle under CE and ED.

ART. 156. *If from a point without a circle a secant and a tangent be drawn to the circle, the square of the tangent is equal to the rectangle under the whole secant and its external part.*

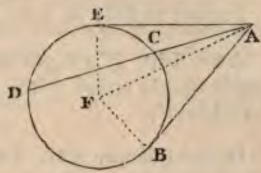


Let AB be a tangent, and AD a secant, from the point A to the circle DBC. Then the square of AB is equal to the rectangle under AD and AC.

DEM. Draw two right lines joining the point of contact B with the points C and D where the secant meets the circumference. By ART. 75, the angle ABC, under the tangent AB, and the chord BC, is equal to the angle BDC in the alternate segment. Therefore, as the angle at A is common to the two triangles ABC, ABD, these triangles are equi-angular to each other, by ART. 39.—Hence, by ART. 112, as AD is to AB, so is AB to AC; that is, by ART. 102, the rectangle under AD and AC is equal to the square of AB.

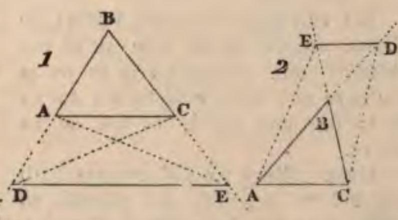
ART. 157. *If from a point without a circle two right lines be drawn, one cutting the circle, the other meeting it at any point, and if the rectangle under the whole secant and its exterior part be equal to the square of the line which meets the circle,—then, this line is a tangent.*

Let AD be a secant drawn from the point A; and AB a right line from the same point meeting the circle in B. Also let the rectangle under AD and AC be equal to the square of AB. Then AB is a tangent to the circle DBC.



DEM. Draw from the point A a tangent AE, and join the centre F with the points E and B. By the preceding ART., the square of AE is equal to the rectangle under AD and AC, which rectangle is granted equal to the square of AB; consequently the square of AE is equal to the square of AB, and therefore, by ART. 34, AE is equal to AB. In the triangles AFE, AFB, the three sides of the one are, therefore, equal to the three sides of the other, respectively; as FE and FB are radii of the same circle. The angle at B is consequently equal to the angle at E, by ART. 6; but the angle at E is equal to a right angle, by ART. 56, and therefore so is the angle at B.—Hence, by ART. 55, as AB is perpendicular to the radius FB, it is a tangent.

ART. 158. *A right line parallel to any side of a triangle, and meeting the other sides produced, cuts them so that the segments of the one have the same ratio as the segments of the other**.



Let the right line DE, parallel to AC, cut the sides BA, BC, produced through A and C, as in fig. 1, or both produced through B, as in fig. 2. The segments of AB, are DA, DB, and the segments of BC, are EC, EB. Then, DA is to DB as EC to EB.

DEM. The proof is exactly the same as in ART. 110.

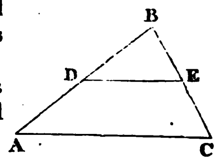
* This is merely an extension of ART. 110.

ART. 159. *A right line cutting any two produced sides of a triangle, so as to make the segments of the one proportional to the corresponding segments of the other, is parallel to the third side.*

Proved exactly as ART. 111.

ART. 160. *If a right line parallel to any side of a triangle divides either of the other sides equally, it will divide both equally.*

In the triangle ABC, if the parallel DE divides AB equally at D, it divides BC likewise equally at E.



DEM. By ART. 110, AD is to DB as EC to EB; therefore as AD is equal to DB, EC is equal to EB.

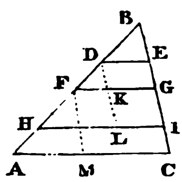
ART. 161. *If a right line divide two sides of a triangle equally, it is parallel to the third.*

If in the above figure DE divide the sides AB, BC, equally at the points D and E, then DE is parallel to AC.

DEM. By ART. 111.

ART. 162. *If several right lines be drawn parallel to a side of any triangle, and meeting the other sides, the segments of one of these sides have the same ratio as the corresponding segments of the other.*

Let the right lines DE, FG, HI, be drawn parallel to the side AC of the triangle ABC. Then BD is to DF as BE is to EG; and DF is to FH as EG is to GI; and FH is to HA as GI to IC.



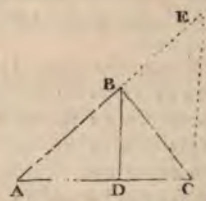
DEM. Since DE and FG are both parallel to AC, they are parallel to each other, by ART. 127.—Hence, by ART. 110, since in the triangle FBG, DE is parallel to FG, BD is to DF as BE is to EG.

Again: Draw DL parallel to BC; and KE, LG, will be parallelograms, by ART. 127 and DEF. XI. Consequently DK is equal to EG, and KL to GI, by ART. 20.—Hence, as

in the triangle HDL, DF is to FH as DK is to KL, by ART. 110, therefore DF is to FH as EG is to GI.

In the same manner, by drawing FM parallel to BC, it may be shown that FH is to HA as GI to IC.

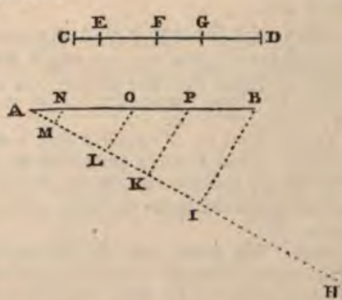
ART. 163. *A right line dividing any angle of a triangle into two equal parts, divides the opposite side into segments which have the same ratio as the sides which contain the divided angle.*



Let ABC be a triangle, of which the angle at B is divided into two equal parts by the right line BD. Then AD is to DC as AB is to BC.

DEM. Draw CE parallel to BD and meeting AB produced in E. By ART. 14, the angle BEC is equal to the angle ABD, or to the angle DBC; and by ART. 12, the angle ECB is equal to the angle DBC; therefore the angle BEC is equal to the angle ECB, and therefore, by ART. 5, BE is equal to BC. But by ART.

PROB. XXV. *To divide a given right line into any number of parts which shall have the same ratio to each other as the parts of another given divided right line.*



Let CD be a right line divided into any number of parts at the points E, F, G. It is required to divide the line AB into the same number of parts, and so that AN may be to NO as CE to EF, and NO to OP as EF to FG, and OP to PB as FG to GD.

CONS. From the point A, draw a right line AH, making an angle with AB. Upon this line AH take AM, ML, LK, KI, equal respectively to CE, EF, FG, GD. Join IB, and draw KP, LO, MN, all parallel to IB. Then the line AB is divided at the points, N, O, P, as required.

DEM. By Art. 162.

110, AD is to DC as AB to BE.—Hence, AD is to DC as AB to BC.

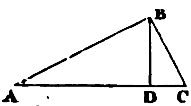
ART. 164. *If a right line drawn from any angle of a triangle to the opposite side divide that side into parts which have the same ratio as the corresponding sides about he given angle, this angle is divided into two equal parts.*

In the above figure, let the right line BD be so drawn that AD may have to DC the same ratio as AB to BC*. Then, the angle ABC is divided into two equal parts ABD and DBC.

DEM. By ART. 110, AB is to BE as AD to DC, and therefore as AB to BC, by the terms of the present ART. Consequently BE is equal to BC, and therefore (ART. 4) the angle BCE to the angle BEC. But as BD is parallel to EC, the angle DBC is equal to BCE, and the angle ABD to BEC, by ARTS. 12 and 14.—Hence the angles DBC and ABD are equal. [NOTE WW.]

ART. 165. *In a right-angled triangle, a perpendicular drawn from the vertex of the right angle to the opposite side, is a mean proportional between the segments of this side.*

Let ABC be a triangle, right angled at B; and let BD be perpendicular to AC. Then AD is to BD as BD to DC.



DEM. As the angle at A is common to the two triangles ABD and ABC, and as the angle ABC is equal to the angle ADB, by ART. 8,—the angle ACB is also equal to the angle

PROB. XXVI. *To make a square equal to a given rectangle.*

CONS. Find a mean proportional between any two adjacent sides of the given rectangle, by PROB. XX. Then a square described upon the mean so found will be equal to the given rectangle.

DEM. By ART. 102; for the square of the mean found, is equal to the rectangle under the extremes, (that is, under the two adjacent sides of the given rectangle), or to the given rectangle itself. This, &c.

* The assertion made in this theorem is true only when AB is to DC as the *corresponding* side AB is to the corresponding side BC. We use the word "corresponding" as plainer, though not so adequate, as "conterminous," for which see p. 125.

ABD (ART. 39). In the same way the angle CBD is equal to BAD. Consequently, the triangles ABD, CBD, are equi-angular to each other.—Hence, the corresponding sides have the same ratio to each other, by ART. 112; that is, AD is to DB as DB to DC.

ART. 166. *In such a triangle as above described, each side about the right angle is a mean proportional between the corresponding segment and the side opposite the right angle.*

In the above figure AD is to AB as AB to AC; also CD is to CB as CB to CA.

DEM. Since the triangles ABD, ABC, have the three angles of the one respectively equal to the three angles of the other, consequently, by ART. 112, the corresponding sides are proportionals.—Hence, AD is to AB as AB to AC. In the same manner CD is to CB as CB to CA.

ART. 167. *In such a triangle as above described, the three sides and the perpendicular are proportionals.*

In the above figure, AC is to AB as BC to BD.

DEM. The triangles ABC, ABD, are equi-angular to each other.—Hence, by ART. 112, AC is to AB as BC to BD.

ART. 168. *In the same circle, the angles, whether at the centre or circumference, have the same ratio to each other as the arches they stand on.*

Proved in the same manner as ARTS. 119 and 120.

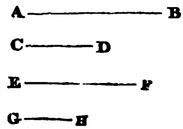
ART. 169. *In a circle, any angle is to four right angles as the arch on which it stands to the whole circumference.*

In the circle ACD, the angle ABC is to four right angles as the arch AC to the circumference ACDA.

DEM. Draw BE perpendicular to BA produced to D. Then the arches AE, ED, DF, FA, are equal, by ART. 86, because the angles standing upon them being right angles are equal by ART. 8. But by last ART. the angle ABC is to the angle ABE, as the arch AC to the arch AE.—Hence, the angle ABC is to four times the angle ABE as the arch AC to four times the arch AE, or the whole circumference AEDFA.

Obs. There are several changes which may be performed on a series of four proportionals, and which the student will find useful in practice. It may be well to annex them in the form of Articles to the preceding Collection.

ART. 170. *In a series of four proportional right lines, the second is to the first as the fourth to the third.*



Let AB be to CD as EF to GH. Then also, CD is to AB as GH to EF.

DEM. By ART. 100, the rectangle under CD and EF is equal to the rectangle under AB and GH.—Hence by ART. 101, CD is to AB as GH to EF.

Obs. The change on four proportionals made in the above ART. is called INVERTENDO, or *Inversion*.

ART. 171. *In a series of four proportional right lines, the first is to the third as the second to the fourth.*

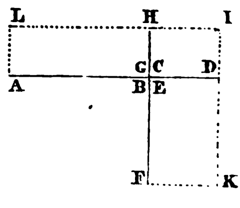
In the above diagram, AB is to EF as CD to GH.

DEM. By ART. 100, the rectangle under AB and GH is equal to the rectangle under CD and EF.—Hence, by ART. 101, AB is to EF as CD to GH.

Obs. The change made in this ART. is called PERMUTANDO, or ALTERNANDO, or *Permutation*, or *Alternation*.

ART. 172. *In a series of four proportional right lines, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

Let AB be to CD as EF to GH. Then the sum of AB and CD is to CD as the sum of EF and GH is to GH.

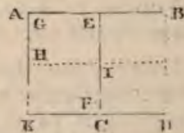


DEM. By Art. 100, the rectangle under AB and GH is equal to the rectangle under CD and EF. Add the rectangle under CD and GH to both sides, and then the whole rectangle AI is equal to the whole rectangle IF. Consequently, by ART. 101, the sides of these rectangles are reciprocally proportional; that is, AD is to HI as HF is to DI. But AD is equal to AD and CD, HI to CD, HF to EF

and GH, and DI to GH.—Hence, the sum of AB and CD is to CD as the sum of EF and GH is to GH.

Obs. The change made in this ART. is called COMPONENDO, or *Composition.*

ART. 173. *In a series of four proportional right lines, the difference between the first and second is to the second as the difference between the third and fourth is to the fourth.*

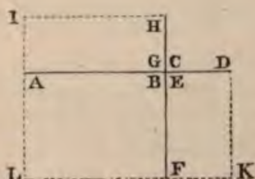


Let AB be to CD as EF to GH. Then the difference between AB and CD is to CD as the difference between EF and GH is to GH.

DEM. By ART. 100, the rectangle under AB and GH is equal to the rectangle under CD and EF. Take away from both the common part BI, and the rectangle AI remains equal to the rectangle ID. Consequently, by ART. 101, the sides of these rectangles are reciprocally proportional; that is, AE is to CD as CI to IE. But AE is equal to the difference between AB and (BE or) CD, CI is equal to the difference between EF and (EI or) GH, and IE is equal to GH. Hence, the difference between AB and CD is to CD as the difference between EF and GH to GH.

Obs. The change made in this ART. is called DIVIDENDO, or *Division.*

ART. 174. *In a series of four proportional right lines, the first is to the sum of the first and second as the third is to the sum of the third and fourth.*

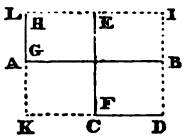


Let AB be to CD as EF to GH. Then, AB is to the sum of AB and CD as EF is to the sum of EF and GH.

DEM. Adding the rectangle AF to the equal rectangles (ART. 100) under AB and GH, EF and CD, the rectangle IF becomes equal to the rectangle AK. Hence, by ART. 101, IH is to AD as DK to HF; that is, AB is to the sum of AB and CD as EF is to the sum of EF and GH.

Obs. The change made in this Art. is called Convertendo, or *Conversion*.

Art. 175. *In a series of four proportional right lines, the first is to the difference between the first and the second as the third is to the difference between the third and the fourth.*



Let AB be to CD as EF to GH. Then AB is to the difference between AB and CD as EF is to the difference between EF and GH.

Dem. From the equal rectangles (Art. 100.) under AB and GH, EF and CD, taking away the common part EB, there remains GE equal to FB. Adding to those the rectangle AC, then the whole HC is equal to the whole KB.—Hence, by Art. 101, AB is to KC as KH to KA; that is, AB is to the difference between AB and CD as EF is to the difference between EF and GH. [Note XX.]

Obs. The change made in this Art. is also called Convertendo, or *Conversion*.

NOTES TO PART III.

Note HH. Simson leaves out the word "integral," which renders his definition inaccurate, and adopts the word "measure," which renders it nearly unintelligible.

Note II. It is with the utmost diffidence that we venture to dissent from the opinion of Dr. Barrow, and indeed all the great geometers who have followed him, that this Definition in the Elements of Euclid is nugatory and superfluous. How are we to understand the word when it occurs in the following definition, if it be not explained here? In Def. V. El. of Euclid, the phrase "*same* ratio" is explained, but not the word ratio; so that if the present definition were left out, as Dr. B. insinuates it might be, we should in Def. V. define the meaning of *two* words, while *one* of them remained unintelligible. It would be the same thing as if we defined what *equal* triangles were, without first defining what triangles themselves were. As for Dr. B.'s objection that this definition is not repeated in Book 7th, where it was as necessary as here,—it may be asked, why should the same thing be defined *twice*? And as to his other remark that nothing is deduced from it, he might as well say that the general definitions of an Angle, a Figure, &c., were useless, because neither from these is any thing deduced. The fact is, that this is a definition of *abstract* Ratio, which is the mutual relation of two similar quantities as to magnitude; and Def. V. regards not this at all, but quite a different thing, *viz.*, identical or *same* ratio,—that is, a particular kind of ratio.

Quantities are of three kinds, Number, Time, and Extension. Any two, of the same kind, may be compared as to greatness or magnitude, and their relation in this respect is called their Ratio. The common definition of Ratio would seem to exclude all quantities but *geometrical* from the possibility of having this relation, which is equally incident to them all.

Note KK. For this definition of four Proportionals we are indebted to a work published in Ireland by Dr. Elrington, the present Bishop of Ferns *. Euclid's definition has long been considered objectionable, inasmuch as it has no resemblance to the common notion of proportionality. It is acknowledged, on all hands, that definitions should be taken from the easier and more familiar notions with which we are in general conversant. At all events, for a treatise like the present, which is intended to

* *The First Six Books of the Elements of Euclid.* Dublin.

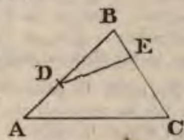
be elementary and popular, the definition given in Euclid is totally unsuitable. Remote as it is from common principles, we could never hope to explain it, nor even to illustrate it, by any examples. In choosing the definition in our text, we have been able to bring down this, the most abstract doctrine of Plane Geometry, to the level of all capacities, by showing how the common idea of proportional magnitudes gradually refines itself into the scientifical definition.

Let us now proceed to show the adequacy of this Definition, and explain its nature more fully. It is to be particularly observed, that where two quantities have the same ratio to each other as two other quantities have, *every* submultiple of the first must be contained the same integral number of times (and no oftener) in the second that an equi-submultiple of the third is contained in the fourth. Thus, 6 has to 42 the same ratio as 9 to 63; because the submultiples 1, 2, 3, (and *all others*), of 6, are contained in 42 the same integral number of times that the *equi*-submultiples $1\frac{1}{2}$, 3, $4\frac{1}{2}$, (and all corresponding ones) of 9, are contained in 63. But 8 has *not* to 13 the same ratio as 16 to 27; because, although certain submultiples of 8, as 1, 2, 4, are contained in 13 the same integral number of times that the equi-submultiples of 16, i. e. 2, 4, 8, are contained in 27, yet a submultiple of 8, viz. $\frac{1}{2}$, is contained 26 times in 13, while the equi-submultiple of 16, viz. 1, is contained 27 times in 27: hence, the ratio of 8 to 13 is not the same as that of 16 to 27. Again: We are not to understand by the terms of this Definition that, when two quantities have the same ratio as two others, every submultiple of the first is contained in the second; but merely that for every submultiple of the first which happens to be contained in the second, there is an equi-submultiple of the third contained as often in the fourth. Thus 42 has to 6 the same ratio as 63 to 9. Because, though 7, which is a submultiple of 42, is not contained at all in 6, neither is the equi-submultiple $10\frac{1}{2}$ of 63 contained in 9; but 1, 2, and every other submultiple of 42 *which is* contained in 6, is contained in it the same integral number of times that $1\frac{1}{2}$, 3, and every other corresponding submultiple of 63 is contained in it. Also: By comparing this Definition with Def. XXXIII. the reader will perceive that there is a distinction between "contained" and "contained *exactly*." Thus 2 is contained 3 times in 7, but not 3 times exactly. If we had introduced the word "exactly" into the present definition, we should have thereby excluded a very large and important class of quantities from the possibility of being compared as to ratio in our sense of the term. For example, there is no portion, however small, of the side of a square which is contained an exact number of times in the diagonal: yet the side and diagonal of a square may have the same ratio as two other quantities. In order that our definition should embrace such quantities, we purposely omit the word "exactly;" and require only that every submultiple of the first of four quantities should be contained in the second the same integral number of times (though with a remainder), that an equi-submultiple of the third is contained in the fourth (though with a remainder). It would be needless to push this subject farther in a popular and elementary Treatise. We have said enough to show that the terms of our definition render it applicable to *incommensurable* quantities, i. e. those which have no common submultiple, or *measure*.

NOTE LL. The connexion between the Doctrine of Rectangles and of Numbers will be shown in our treatise on the application of ALGEBRA to GEOMETRY. In the present work, it is our design to restrict the learner as far as possible to ideas purely geometrical, although the doctrine of proportion being to a certain degree arithmetical, we cannot help at times speaking of number.

NOTE MM. All the Articles in this LESSON may be proved in the same manner as ARTS. 94, 95, 96, 97, 98, 99, respectively.

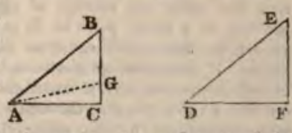
NOTE NN. In the enunciation of this Article it is necessary to introduce the condition—that the corresponding segments should be at the same side as the intersector; for, if DA were to DB as BE to EC, the line DE would not be parallel to AC*.



NOTE OO. This proof is slightly different from that in the first Edition, owing to the change made in ART. 18.

NOTE PP. It is plain at once from ARTS. 8 and 39, that if the third angles be both *right* ones, the triangles must be equiangular.

The triangles are also equiangular if, an angle in the one being equal to an angle in the other, and the sides about another pair of angles proportional,—*one* of the remaining angles is given a right angle. Thus, if the angle at B be equal to the angle at E, and if BA be to ED as AC to DF, then the triangles are equiangular if but one of the angles at C and F is given a right angle. For, let the angle at C be the right angle, and *suppose* the angle BAC not equal to EDF, but that the angle BAG is equal to the angle EDF. Then, in the same way as in the demonstration of ART. 115, AG would be proved equal to AC, and consequently the angles at C and G equal, by ART. 4, and therefore both of them acute, by ART. 38. But the angle at C is granted to be a right one; consequently the above supposition is false,—that is, the angle BAG is not equal to EDF. And in the same way it can be shown that no other angle but BAC is equal to EDF;—hence, the triangles ABC, DEF, are equiangular, by ART. 39.



NOTE QQ. This may be demonstrated in the same manner as the preceding Article.

NOTE RR. With all the advantages of our definition of Proportionals, in point of clearness and familiarity, above Euclid's, the latter has this one circumstance to recommend it, that it affords a *direct* proof of the present Article, instead of the indirect one which we are obliged to make use of. Such a superiority is, however, much more than compensated by the general ease and satisfaction with which a reader is enabled to master the Doctrine of Proportion as built upon our Definition.

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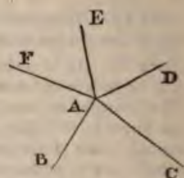
* Unless DA happened to have to DB a *ratio of equality*; that is, unless DA and DB happened to be equal.

Dr. Elrington, in his Euclid before mentioned, gives a direct proof of this Article, founded on the definition of proportionals, which we have adopted. But this proof is not perfectly legitimate, involving a problem never yet solved, namely, the trisection of an angle. For our own part, we think that this circumstance detracts nothing from the legitimacy of Dr. Elrington's proof; though mathematicians have not found out a geometrical construction for the trisection of an angle, the problem is evidently *possible*, and all results grounded on the *supposition* of the problem being resolved are to us as conclusive as if it were actually resolved. But we have thought it best to control our private opinion, and rather to give an indirect proof, than a direct one, whose legitimacy was, in the slightest degree, questionable.

NOTE SS. In geometry we often speak of *the angles round a point*. By this we mean all the angles which can be formed by lines diverging from a point in space.



Thus the angles BAC, CAD, DAE, EAF, FAB, are the angles round the point A.

It is plain, from ART. 125, that all the angles round a point are together equal to four right angles.

We also frequently speak of an angle equal to two right angles. This is confessedly an impropriety of speech, but it is recognised for brevity's sake. All the angles which can be made at one side of a right line AB, by any number of right lines meeting at a given point in it, as CD, CE, CF, are, by ART. 123, equal to two right angles; if, therefore, we choose to consider CA and CB the two sides of an angle spread out into a right line, then the whole angular space (measured by the semicircle *m n o*) at the point C contained between CA and CB, is equal to two right angles.

NOTE TT. This is the second part of PROP. Vth., B. I., EUCLID. But it is of no great utility, and therefore we have retained the first part only (ART. 4) in our text.

NOTE UU. There is little difficulty in giving a proof of this theorem, on our principles, but we had inserted an insufficient one in the first Edition. Those who may have thought our system of parallels invalid from this circumstance, will be undeceived by the above demonstration. We have not the slightest wish to conceal any defect which may exist in our theory; but it certainly does not lie here. Our *definition* of parallels contains perhaps more than we are mathematically entitled to assume; but it is, at all events, sufficiently accurate in a popular view, being no more than the popular notion of parallels, scientifically stated.

It is to be observed that the above theorem not being requisite to our Elements may be proved by *any* of them, although we have thought fit to prove it here, lest invalidity might have been presumed as the cause of our postponing it. The following is a very simple demonstration from the principles of PART III.

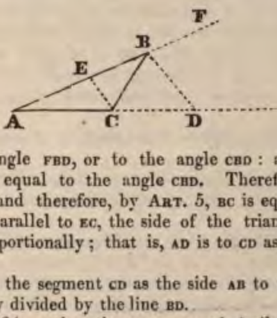
Take AU a fourth proportional to AK, AB, and AI, (by PROB. XVIII). Then BU is parallel to the right line joining IK. But BD is also parallel to the same line, and therefore must coincide with BU (ART. 13). Hence, BD meets AF in the point U.

We have only to add, that this is the proposition which is assumed by Euclid as *self-evident*—the famous 12th Axiom. Upon it is founded almost the whole System of Geometry as now studied. The reader will perhaps grant that a treatise, such as the common Euclid, which sets out with so monstrous and unwarrantable an assumption, needs some alteration. It is but just to acknowledge, however, that the system of Euclid would be much less objectionable, (as MONTUCLA notices,) if the 12th Axiom, instead of being assumed in the threshold of the Elements, had been assumed after PROP. 28th, of the latter part of which it is, in fact, the converse.

NOTE VV. ART. 148, which in Euclid is deduced as a particular result from PROP. V., B. II., is evidently the general theorem under which PROPS. V. and VI. are contained.

NOTE WW. The assertions made in this and the preceding Articles are true, when the *external* angle of triangles is divided equally. Thus:

In the triangle ABC, let the external angle CBF be divided into two equal parts by BD. Then, the segment AD is to the segment CD as the side AB is to the side BC.



DEM. Draw CE parallel to BD, and meeting AB in E. By ART. 14, the angle BEC is equal to the angle FBD, or to the angle CBD: and by ART. 12, the angle BCE is also equal to the angle CBD. Therefore the angles BEC and BCE are equal, and therefore, by ART. 5, BC is equal to BE. But by ART. 158, as BD is parallel to EC, the side of the triangle ACE, it cuts the sides AC and AE proportionally; that is, AD is to CD as AB is to (BE or) BC.

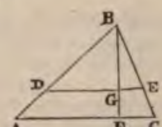
Again:—If the segment AD is to the segment CD as the side AB to the side BC, then the angle CBF is equally divided by the line BD.

This is proved similarly to ART. 164, as the other was proved similarly to ART. 163.

NOTE XX. An instance of the utility of these changes upon four proportionals may be seen in the following theorem.

If a right line parallel to any side of a triangle be drawn, and another from the vertex of the opposite angle meeting the third side and the parallel, then the segments of the third side have the same ratio to each other as the segments of the parallel.

In the triangle ABC let DE be parallel to AC, and BF be any line from the vertex B meeting AC and DE in the points F and G. Then AF is to FC as DG to GE.

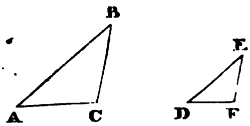


DEM. DG being parallel to AF, the triangles DGB, ABF, are equiangular; and therefore, by ART. 112, AF is to DG as FB to GB. In the same way FC is to GE as FB to GB.

Consequently AF is to DG as FC to GE; hence, by ART. 171, AF is to FC as DG to GE.

By the help of ART. 172, we may prove that DA is to the whole side BA as CE to the whole side BC; for, by ART. 110, DA is to DB as EC to EB, and therefore DA is to the sum of DA and DB as EC to the sum of EC and EB.

By the help of ART. 171, we can show that in the equiangular triangles ABC, DEF, the sides about any angle in the one have the same ratio as the sides about the equal angle in the other. For, by ART. 112, AB is to DE as AC to DF;—hence, by ART. 171, AB is to AC as DE to DF.



Numberless other cases will occur to the mathematical student in which these Articles will be found extremely useful.

EXPLANATION OF SOME TECHNICAL TERMS MADE USE OF IN GEOMETRY.

Corollary. A corollary is an immediate ***consequence*** of a Proposition. Thus our Section of Results is a Section of Corollaries.

Bisect. To bisect is to divide into ***two*** equal parts (as in PROBS. IV. and V.)

Trisect. To divide into ***three*** equal parts.

Scalene Triangle. A triangle is scalene when ***all*** of its sides are unequal.

Isosceles Do. An isosceles triangle is that which has ***two*** equal sides (as in ART. 4).

Equilateral Do. An equilateral triangle is that which has ***all*** its sides equal (as in ART. 121).

Equiangular Do. An equiangular triangle is that which has ***all*** its angles equal (as in ART. 122). Two triangles are also said to be equiangular ***to each other***, when the three angles of one are respectively equal to the three angles of the other (as in ART. 112, &c.).

Equivalent Triangles. Triangles (or indeed any figures) are said to be equivalent, when they are equal in ***area*** (like those in ARTS. 23, 25, &c.).

Quadrilateral. A quadrilateral figure is a ***four-sided*** rectilineal figure (as in ART. 40).

Trapezium. A trapezium is generally taken for a rectilineal figure of four sides, ***two*** only of which are parallel (as the fig. ADEC, in ART. 160).

Polygon. A polygon is a many-sided rectilineal figure (as in DEF. IV.); and any right line joining the vertices of any two of its angles, so as to divide the figure into two parts, is considered as a ***diagonal*** of the polygon. It is called a ***regular*** or ***ordinate*** polygon, when all its sides are equal.

Subtense. A subtense is a line which stretches ***across*** an angle or arch; (as the line FG in PROB. VI., which stretches across the angle FCG, or the arch FDG, is respectively the subtense of the angle FCG, or of the arch FDG. Also the circular line FDG is the subtense of the angle FCG). A subtense is said to ***subtend*** its angle or arch.

Hypotenuse. An hypotenuse is that side of a right-angled triangle which is opposite to (or subtends) the ***right*** angle (as in ART. 47, AC is the hypotenuse of the triangle ABC).

Conterminous. Conterminous magnitudes are such as have their terminations ***coinciding*** (as in ART. 6, where FC and BC, or ED and BD, are conterminous lines, the terminations of the two former coinciding in the point F, those of the two latter in the point D).

In directum. Magnitudes are said to be ***in directum*** (or ***in continuum***) when they form one right or straight magnitude (as in ART. 10, where BC is ***in directum*** with BD).

Ad absurdum. A demonstration is said to be *ad absurdum* (or *ad impossibile*) when it is of the indirect kind (as in Arts. 5, 10, &c.) In this we demonstrate the truth of our own assertion, by demonstrating the *absurdity* which would follow from *supposing* it false.

Perimeter. The perimeter of a figure is the line, or lines, which contain it.

Periphery. The circumference.

Similar Segments of circles. Segments of circles are said to be similar when they contain *equal* angles (as in Art. 84, fig. 2, the segments AEB, CFD, are similar, because the angles AEB, CFD, are equal).

Antecedents and Consequents. In a series of four proportionals, the *first* and *third* are called antecedents : the *second* and *fourth* consequents. Thus, in Art. 100, the lines AB and EF are the antecedents; the lines CD and GH are the consequents.

Note. A series of four proportionals is often marked in this way, *viz.* A : B : : C : D,—that is to say, *As* A *is to* B *so is* C *to* D. A series of three proportionals is marked thus—A : B : C, *i.e. As* A *is to* B *so is* B *to* C.

Homologous, means *corresponding.* Thus, in Art. 112, the angle ABC is homologous to DEF, the angle BAC to EDF, and the angle ACB to DFE. Thus also in a series of four proportionals, the antecedents are homologous to each other, as also the consequents.

Similar Figures. Rectilineal figures are said to be similar, when the angles of the one are respectively equal to the angles of the other, and the sides about the equal angles proportional. Thus, in Art. 112, or 118, the triangles ABC, DEF, are *similar.* In similar figures the corresponding sides and angles are *homologous.*

A COMPARATIVE TABLE,

Showing what Propositions and Corollaries in Euclid* correspond with the Articles and Problems in this Treatise.

Art.	Book I.
1, 2, 3	correspond to Prop. IV.
4	corresponds to V.
5	. . . . VI.
6	. . VIII.
7	. . . . .
8	. . . . .
9	. . XIII.
10	. . XIV.
11	. . . XV.
12, 13, 14	correspond to XXIX.
15	corresponds to XXVII
16, 17	correspond to XXVIII.
18	. . . . .
19	. XXXIII.
20, 21, 22	correspond to XXXIV.
23	corresponds to XXXV.
24	. XXXVI.
25	. . XXXVII.
26	. XXXVIII.
27	. . XXXIX.
28	. . . . .
29	. . . . .
30	. . . . .
31	. . XLI.
32	. . . . .
33	. . . . .
34	. . . . .
35	Part 1, XXXII.
36	. . XVI.
37	Part 2, XXXII.
38	. . XVII.
39	. . . . .
40	a *Case* of Cor. 1. XXXII.
41	. . XVIII.
42	. . . XIX.
43	. . . XX.
44	. . XXIV.
45	. . XXV.
46	. . XXVI.
47	. . XLVII.
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* R. Simson's is that referred to.

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* The remaining *Articles* have no correspondent propositions in Euclid.

THE END.

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